Assessing Supplier's Process Capability using Truncated Normal Distribution Data

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Abstract: *Process Capability Indices (PCIs) have been proposed to assess process capability in real world application over the past four decades. Today, purchasing personnel also uses PCIs to select best supplier. To select best supplier using supplier's values of PCIs may not be reliable due to fear of data manipulation. To assess supplier's process capability, purchasing personnel needs process distribution with parameters referring received samples from supplier. Most of the time, received samples conform hundred percent to the specification, because products are categorized as conforming and nonconforming by supplier before being sent to customer and only conforming products are sent to customer. If the process distribution is normal then process distribution identified by referring to received sample products is truncated normal. From a customer's point of view intention of this paper is estimation of process distribution parameter referring to truncated normal data by identifying best method of parameter estimation with respect to accuracy and precision of estimates. It is found that method of moments provides best estimators of process parameters without loss of efficiency as compared to other competing methods. Through simulation, performance of method of moments is compared with other competing recently developed methods which include maximum likelihood estimation starting from re-parameterization and quantile-filling algorithm (QA) based on EM (Expectation-Maximization) algorithm. Estimated parameters are used to estimate supplier's process capability using probability based PCIs through illustrative example.*

Keywords: **Process capability indices, supplier's process capability, process distribution, normal distribution, truncated normal distribution.**

1. Introduction

Univariate basic Process Capability Indices (PCIs) C_p, C_{pk} and C_{pm} are developed to measure process capability in manufacturing industries assuming process distribution is normal with process mean μ and process standard deviation σ. While developing the basic PCIs it is also assumed that process is stable and a specification region of a quality characteristic is specified by lower and upper specification limits (LSL and USL) with target value T as (LSL+USL)/2. These PCIs are related with each other but each one measures process capability in a different aspect. The PCIs C_p and C_{pk} measure potential capability and actual capability respectively while C_{pm} assesses process performance taking into account closeness to the target and variability of the process.

In today's marketplace purchasing personnel also uses PCIs to select best supplier amongst several suppliers. To select best supplier using supplier's published values of PCIs may not be reliable due to fear of data manipulation. To assess supplier's process capability, purchasing personnel needs process distribution with parameters referring received sample of products from supplier. Most of the time received sample of products from supplier is hundred percent conforming to the specification, since products are categorized as conforming and nonconforming by supplier before being sent to customer and only conforming products are sent to customer. If we assume process distribution is normal with process mean μ and process standard deviation σ then the process distribution identified on the basis of received sample of products must be either doubly truncated normal or left truncated normal or right truncated normal. In case specification limits of a quality characteristic are specified by lower & upper specification limits (LSL and USL), then identified distribution is doubly truncated normal. In other cases, if quality characteristic is specified by just LSL then identified distribution is left truncated normal and if quality characteristic is specified by just USL then identified distribution is right truncated normal. From a customer's point of view, the problem is how process parameter of the process distribution while referring to truncated data will be estimated. Under normal distribution, estimation of parameters using method of moments and maximum likelihood estimation is not difficult task. This task is difficult when we refer to truncated normal distribution to estimate process parameter.

The intention of this paper is estimation of process distribution parameter to measure supplier's process capability by referring to truncated normal data by identifying best method of parameter estimation with respect to accuracy and precision of estimates. Section 2 devoted to literature review and in section 3, we discuss theoretical background of doubly truncated normal distribution, left truncated normal distribution and right truncated normal distribution. Section 4 deals with parameter estimation using method of moments, method of maximum likelihood estimations, method of maximum likelihood starting from re-parameterization and Quantile-filling Algorithm (QA) based on EM (Expectation-Maximization) algorithm. Section 5 is devoted to simulation study in view to measure performance of method of moments with other competing methods. Section 6, deals with basic PCIs with their alternatives based on probability with illustrative example. The last section presents concluding remarks.

2. Literature Review

PCIs have been studied extensively in the literature with their estimators, distributional and inferential properties (Kotz and Johnson [1], Kotz and Lovelace [2] and references there in). Pearn et al. [3] proposed the PCI C_{pmk} referred to as the 'third generation capability index' which combined the merits of C_p, C_{pk} and C_{pm}. The C_{pmk} is more sensitive to the departure of process mean from target than C_p, C_{pk} and C_{pm}. Polansky et al. [4] estimated process capability for a truncated distribution by applying Johnson transformation. Univariate as well as multivariate process capability indices were recently reviewed by de-Felipe, D. and Benedito, E. [5] with strengths and weaknesses of each index.

A. C. Cohen [6] has estimated parameters of doubly and singly truncated normal distribution using maximum likelihood estimation. A. C. Cohen [7] has also estimated parameters of singly truncated normal distribution using first three sample moments. S. M. Shah and M. C. Jaiswal [8] have used method of moments to estimate the parameters of doubly truncated normal distribution considering first four sample moments. They have also compared method of moments estimator with maximum likelihood estimators proposed by A. C. Cohen [6] and concluded that method of moments provides simple estimators of the parameters without much loss of efficiency, while the maximum likelihood estimators are complicated and laborious. Pueyo S. [9] has given a simple procedure to estimate the parameters of the truncated normal distribution by maximum likelihood starting from re-parameterization. Truncated distribution dataset can be treated as special case of interval-censored data. Many data imputation methods have been developed for interval-censored data to convert it into pseudo-complete data set. Recently, Jun Yang et al. [10] used quantile-filling algorithm (QA) based on EM (Expectation-Maximization) algorithm (QA-EM) to convert the truncated data into pseudo-complete data for parameters estimation. They have also compared their proposed method with other two competing methods.

3. Truncated Distributions

Consider random variable X with distribution function $F(x)$ and p.d.f. $f(x)$. Suppose for some reason we discard the values of X less than 'a' and greater than 'b' provided $a < b$. Thus we are considering those values of X which are in $[a, b]$. In such a situation the resulting distribution of X may be viewed as a conditional distribution subject to the hypothesis $a \le x \le b$. This distribution is called as truncated distribution. Let us denote $F_T(x)$ as the distribution function and $f_T(x)$ as the p.d.f. of truncated distribution.

3.1 Doubly Truncated Normal Distribution

Let $X \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty;\ -\infty < \mu < \infty \ and \ \sigma > 0 \tag{1}$$

$$Mean = E(X) = \mu \ and \ Variance = V(X) = \sigma^2$$

Suppose for some reason, we discard the values of *X* less than '*a*' and greater than '*b*' provided *a* < *b*. In such a situation resulting distribution is doubly truncated normal having probability density function with mean and variance is given below:

$$f_T(x) = \frac{1}{M^* \sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, a \le x \le b;\ M^* = F(b) - F(a) \tag{2}$$

$$Mean = E_T(X) = \mu + \frac{\sigma^2}{M^*}\left(f(a) - f(b)\right)$$

$$Variance = V_T(X) = \sigma^2\left(\frac{(a-\mu)f(a) - (b-\mu)f(b)}{M^*}\right) + \sigma^2 - \left(\frac{\sigma^2}{M^*}\left(f(a) - f(b)\right)\right)^2$$

3.2 Left Truncated Normal Distribution

Let $X \sim N(\mu, \sigma^2)$, if we discard the values of *X* less than '*a*' then resulting distribution is left truncated normal having probability density function with mean and variance is given below:

$$f_T(x) = \frac{1}{\sigma M_0\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, x \ge a;\ M_0 = 1 - F(a) \tag{3}$$

$$Mean = E_T(X) = \frac{\sigma^2}{M_0} f(a) + \mu \ \text{ and } \ Variance = V_T(X) = \frac{\sigma^2}{M_0}(a-\mu)f(a) + \sigma^2 - \left(\frac{\sigma^2}{M_0} f(a)\right)^2$$

3.3 Right Truncated Normal Distribution

If $X \sim N(\mu, \sigma^2)$ and if we discard the values of *X* greater than '*b*' then the resulting distribution is right truncated normal having probability density function with mean and variance is given below:

$$f_T(x) = \frac{1}{M_1 \sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, x \le b,;\ M_1 = F(b) \tag{4}$$

$$Mean = E_T(X) = \mu - \frac{\sigma^2}{M_1} f(b) \ and \ Variance = V_T(X) = -\frac{\sigma^2(b-\mu)}{M_1} f(b) + \sigma^2 - \left(\frac{\sigma^2}{M_1} f(b)\right)^2$$

Note: Mean and variance of normal distribution is μ and σ^2 but all truncated normal distributions mean is not μ and variance is not σ^2.

4. Parameter Estimation Referring Truncated Data

A. C. Cohen [6] has estimated parameters of doubly and singly truncated normal distribution using maximum likelihood estimation. S. M. Shah and M. C. Jaiswal [8] have used method of moments to estimate the parameters of doubly truncated normal distribution considering first four sample moments. They have concluded that method of moments provides simple estimators of the parameters without much loss of efficiency, while the maximum likelihood estimators are complicated and laborious. Pueyo S. [9] has estimated the parameters of the truncated normal distribution by maximum likelihood starting from re-parameterization. Jun Yang et al. [10] used algorithm QA-EM to convert the truncated data into pseudo-complete data for parameters estimation.

4.1 Method of Moments Estimator Proposed by Shah and M. C. Jaiswal [8]

Considering equation (2), p.d.f. of doubly truncated normal distribution let us define

$\varphi(t) = \frac{1}{\sqrt{2\pi}} e^{(-1/2)t^2}$ and $\Phi(t) = \int_{-\infty}^{t} \phi(x)dx$;

$k_0 = \frac{a-\mu}{\sigma}$, $k_1 = \frac{b-\mu}{\sigma}$ $and \ d = b - a$

Transforming x into y using $y=x-a$, moments m_r of the distribution of y about its origin are given by

$$m_1 = \sigma(z_0 - z_1) - \sigma k_0$$

$$m_r = -\sigma d^{r-1} z_1 + (r-1)\sigma^2 m_{r-2} - k_0 \sigma m_{r-1} \ ; \quad r = 2,3,4,....... \tag{5}$$

$$where \ z_0 = \frac{\phi(k_0)}{[\Phi(k_1) - \Phi(k_0)]} \quad and \quad z_1 = \frac{\phi(k_1)}{[\Phi(k_1) - \Phi(k_0)]}$$

Considering equation (5) when $r=2, r=3,$ and $r=4$ in three unknowns σz_1, $\sigma^2 \ and \ h = \sigma k_0$ and solving them the solutions for $h \ and \ \sigma^2$ are

$$h = \frac{(3m_2 m_3 - 2m_1 m_4) + d(m_4 - 3m_2^2) + d^2(2m_1 m_2 - m_3)}{P} \tag{6}$$

$$\sigma^2 = \frac{(m_3^2 - m_2 m_4) + d(m_1 m_4 - m_2 m_3) + d^2(m_2^2 - m_1 m_3)}{P} \tag{7}$$

$$where \ P = (2m_1 m_3 - 3m_2^2) + d(3m_1 m_2 - m_3) + d^2(m_2 - 2m_1^2)$$

(8)

Replacing m_r with the corresponding sample moments $v_r = \frac{\sum (x-a)^r}{n}$ in equation (6), (7) and (8) we get method of moments estimators $h^* \ and \ \sigma^{2*}$ of $h \ and \ \sigma^2$. The estimator μ^* of μ is given by $\mu^* = a - h^*$.

Remark 1: When $b = \infty$, that is left truncated normal distribution having p.d.f. is given in equation (3) the estimators of $h \ and \ \sigma^2$ are

$$h^* = \frac{2v_1 v_2 - v_3}{v_2 - 2v_1^2} \text{ and } \sigma^{2*} = \frac{v_2^2 - v_1 v_3}{v_2 - 2v_1^2} \tag{9}$$

Remark 2: When $a = -\infty$, that is right truncated normal distribution having p.d.f. is given in equation (4) the estimators are same as equation (9) but in this case $y=x-b$. Here the odd moments negative due to the choice of origin.

4.2 Method of Maximum Likelihood Estimator Proposed by A. C. Cohen [6]

The maximum likelihood estimators $\hat{h}$ and $\hat{\sigma}$ of h and σ of doubly truncated normal distribution obtained by A. C. Cohen [6] in the notation used above are as follows:

$$v_1 = \hat{\sigma}(z_0 - z_1) - \hat{h}$$

$$v_2 = -\hat{\sigma} d z_1 + \hat{\sigma}^2 - \hat{h} v_1$$

Estimators $\hat{h}$ and $\hat{\sigma}$ are to be found by numerical methods because z_0 and z_1 depends on $\hat{h}$ and $\hat{\sigma}$.

4.3 Method of Maximum Likelihood Estimator Proposed by Salvador Pueyo [9]

Equation (2), p.d.f. of double truncated normal distribution can be written as

$$f_T(x) = \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2} du}, \qquad a \le x \le b;$$

$$= \frac{e^{-\alpha x - \psi x^2}}{\int_a^b e^{-\alpha u - \psi u^2} du} \tag{10}$$

$$\text{where } \alpha = -\frac{\mu}{\sigma^2} \ and \ \psi = \frac{1}{2\sigma^2} \tag{11}$$

By definition, the m.l.e. is obtained by maximizing $\Lambda = \sum_{i=1}^{n} \ln f(x_i | \alpha, \psi)$.

In this case, this gives

$$\begin{cases} E(X|\alpha,\psi) = w_1 \\ E(X^2|\alpha,\psi) = w_2 \end{cases} \tag{12}$$

where w_r is the r^th^ sample raw moments of X, i.e. $w_r = \frac{\sum_{i=1}^{n} x_i^r}{n}; r = 1,2,3,......$

Equation (12) does not have closed form, so the estimators have to be obtained iteratively.
Considering following notations for the update of the parameters at step $j+1$ ($j = 1,2,...$)

$$\begin{cases} \alpha_{j+1} = \alpha_j + \Delta_j \alpha \\ \psi_{j+1} = \psi_j + \Delta_j \psi \end{cases} \tag{13}$$

After mathematical derivation the updating rule suggested by Salvador Pueyo [9] is given below

$$\begin{cases} \Delta_j \alpha = a\eta(w_1 - E(X| \alpha_j, \psi_j)) + b\eta(w_2 - E(X^2| \alpha_j, \psi_j)) \\ \Delta_j \psi = b\eta(w_1 - E(X| \alpha_j, \psi_j)) + c\eta(w_2 - E(X^2| \alpha_j, \psi_j)) \end{cases} \tag{14}$$

$$\text{where } \begin{cases} a = (w_4 - w_2^2)/ f_1 \\ b = (-w_3 + w_1 w_2)/ f_1 \\ c = (w_2 - w_1^2)/ f_1 \end{cases} \tag{15}$$

$$\text{and } f_1 = w_4(-w_2 + w_1^2) + w_3(w_3 - 2w_1 w_2) + w_2^3 \tag{16}$$

The expectations in equation (14) are calculated from equation (10), using numerical integration. The value η is relatively arbitrary. The larger the chosen η, the quicker the convergence if it converges, but also the larger the risk that it does not. The author uses $\eta = 0.33$, in case of overflow η should be reduced. The procedure should proceed until $\Delta_j \alpha \ and \ \Delta_j \psi$ becomes smaller than some given thresholds. From estimated values of $\alpha \ and \ \psi$, we can obtain m.l.e. of $\mu \ and \ \sigma^2$ using equation (11).

4.4 Quantile Filling Algorithm based on the Expectation Maximization Method (QA-EM) by Jun Yang et al. [10]

This method transforms truncated normal data considering as incomplete data into pseudo-complete data, which consists of mainly two steps.

EM estimation:

Suppose discarded values of X which are less than 'a' are denoted by $x_{a,1}, x_{a,2}, ..., x_{a,n1}$; undiscarded values of X which lie between [a, b] are denoted by $x_{T,1}, x_{T,2}, ..., x_{T,n2}$ and discarded values of X which are greater than 'b' are denoted by $x_{b,1}, x_{b,2}, ..., x_{b,n3}$. Let $x_a = (x_{a,1}, x_{a,2}, ..., x_{a,n1})$, $x_T = (x_{T,1}, x_{T,2}, ..., x_{T,n2})$ and $x_b = (x_{b,1}, x_{b,2}, ..., x_{b,n3})$ denote the three subsets where $n1, n2, n3$ respectively represent the sample sizes of the three subsets. The complete sample data set can be written as $x = (x_1, x_2, ..., x_n) = (x_a; x_T; x_b)$ where $n = n1 + n2 + n3$.

Based on the complete sample data set log-likelihood function becomes

$$\ln L(\theta) = -n\ln(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 - \frac{n}{2}\ln(2\pi) \tag{17}$$

Since in practice the customers get only observed subset $x_T = (x_{T,1}, x_{T,2}, \ldots, x_{T,n2})$ to estimate $\theta = (\mu, \sigma)$, Jun Yang et al. [10] proposed EM algorithm for the estimation. Let $\theta^{(i)} = (\mu^{(i)}, \sigma^{(i)}), n1^{(i)}$ *and* $n3^{(i)}$ respectively represent the estimators of θ, $n1$ *and* $n3$ after the i^{th} iteration of EM algorithm, *i=1,2,....* From the observed subset $x_T = (x_{T,1}, x_{T,2}, \ldots, x_{T,n2})$ estimate $\theta^{(1)} = (\mu^{(1)}, \sigma^{(1)})$, using MLE as the initial parameter values of the EM algorithm. For *i=1*,

$$n1^{(i)} = \frac{n2\Phi\left(\dfrac{a-\mu^{(i)}}{\sigma^{(i)}}\right)}{\Phi\left(\dfrac{b-\mu^{(i)}}{\sigma^{(i)}}\right) - \Phi\left(\dfrac{a-\mu^{(i)}}{\sigma^{(i)}}\right)} \tag{18}$$

$$n3^{(i)} = \frac{n2\left(1-\Phi\left(\dfrac{b-\mu^{(i)}}{\sigma^{(i)}}\right)\right)}{\Phi\left(\dfrac{b-\mu^{(i)}}{\sigma^{(i)}}\right) - \Phi\left(\dfrac{a-\mu^{(i)}}{\sigma^{(i)}}\right)} \tag{19}$$

where $\Phi(.)$ is distribution function of *N(0,1)*.

The $\theta^{(i+1)} = (\mu^{(i+1)}, \sigma^{(i+1)})$ can be derived by maximizing $E(\ln L(\theta) \mid \theta^{(i)}, x_T)$ which can be computed as follows

$$\mu^{(i+1)} = \frac{\sum_{j=1}^{n2} x_{T,j} + (n1^{(i)} + n3^{(i)})\mu^{(i)} - n1^{(i)}\dfrac{\sigma^{(i)}\phi((a-\mu^{(i)})/\sigma^{(i)})}{\Phi((a-\mu^{(i)})/\sigma^{(i)})} + n3^{(i)}\dfrac{\sigma^{(i)}\phi((b-\mu^{(i)})/\sigma^{(i)})}{1-\Phi((b-\mu^{(i)})/\sigma^{(i)})}}{n1^{(i)} + n2 + n3^{(i)}}$$
$$+ \frac{n3^{(i)}}{n1^{(i)} + n2 + n3^{(i)}} \times \frac{\sigma^{(i)}\phi((b-\mu^{(i)})/\sigma^{(i)})}{1-\Phi((b-\mu^{(i)})/\sigma^{(i)})} \tag{20}$$

$$(\sigma^{(i+1)})^2 = \frac{1}{n1^{(i)} + n2 + n3^{(i)}}\left\{\sum_{j=1}^{n2}(x_{T,j} - \mu^{(i+1)})^2\right.$$
$$+ n1^{(i)}\left[(\sigma^{(i)})^2 + (\mu^{(i)} - \mu^{(i+1)})^2 + (3\mu^{(i)} - 2\mu^{(i+1)} - a)\frac{\sigma^{(i)}\phi((a-\mu^{(i)})/\sigma^{(i)})}{\Phi((a-\mu^{(i)})/\sigma^{(i)})}\right]$$
$$\left. + n3^{(i)}\left[(\sigma^{(i)})^2 + (\mu^{(i)} - \mu^{(i+1)})^2 + (2\mu^{(i+1)} - 3\mu^{(i)} + b) \times \frac{\sigma^{(i)}\phi((b-\mu^{(i)})/\sigma^{(i)})}{1-\Phi((b-\mu^{(i)})/\sigma^{(i)})}\right]\right\} \tag{21}$$

where $\phi(.)$ is the p.d.f. of *N(0,1)*.

The proof of equation (20) and (21) are given by Jun Yang et al. [10]. The iteration will stop if $|\theta^{(i+1)} - \theta^{(i)}| < \varepsilon$; where ε is a precision parameter. $\hat{\theta} = (\hat{\mu}, \hat{\sigma}) = (\mu^{(i+1)}, \sigma^{(i+1)})$ is the final estimator of the EM algorithm.

Quantile filling Algorithm:

The quantile-filling algorithm generate the pseudo-complete data through multiple iterations (*k=1,2,...*). First iteration require initial values of $\theta = (\mu, \sigma)$. Estimated values of $\theta = (\mu, \sigma)$ obtained by EM algorithm are taken as the initial values, that is $\tilde{\mu}^{(0)} = \hat{\mu}$ *and* $\tilde{\sigma}^{(0)} = \hat{\sigma}$. Starting from *k=1* each iteration requires following steps.

1. Determine $n1^{(k)}$ and $n3^{(k)}$ using

$$n1^{(k)} = \frac{n2F_{(k-1)}(a)}{F_{(k-1)}(b) - F_{(k-1)}(a)} \tag{22}$$

$$n3^{(k)} = \frac{n2\left(1 - F_{(k-1)}(b)\right)}{F_{(k-1)}(b) - F_{(k-1)}(a)} \tag{23}$$

where $F_{(k-1)}(x) = P(X \le x);\ X \sim N\left(\mu = \tilde{\mu}^{(k-1)}, \sigma = \tilde{\sigma}^{(k-1)}\right)$

2. The pseudo samples generated using

$$x_{a,j}^{(k)} = F_{(k-1)}^{-1}\left(\frac{jF_{(k-1)}(a)}{n1^{(k)}+1}\right),\ j = 1,2,\ldots,n1^{(k)} \tag{24}$$

$$x_{b,j}^{(k)} = F_{(k-1)}^{-1}\left(F_{(k-1)}(b) + \frac{j\left(1 - F_{(k-1)}(b)\right)}{n3^{(k)}+1}\right),\ j = 1,2,\ldots,n3^{(k)} \tag{25}$$

Where $F_{(k-1)}^{-1}(.)$ is the inverse function of $F_{(k-1)}(.)$

Pseudo complete data at the kth iteration is $x^{(k)} = (x_1, x_2, \ldots, x_{\tilde{n}^{(k)}}) = (x_a^k; x_T; x_b^k);$

where $\tilde{n}^{(k)} = n1^{(k)} + n2 + n3^{(k)}$

3. Estimates of $\theta = (\mu, \sigma)$ using pseudo complete data by MLE are as follows:

$$\tilde{\mu}^{(k)} = \frac{1}{\tilde{n}^{(k)}}\sum_{i=1}^{\tilde{n}^{(k)}} x_i^{(k)}\ and\ \tilde{\sigma}^{(k)} = \sqrt{\frac{1}{\tilde{n}^{(k)}}\sum_{i=1}^{\tilde{n}^{(k)}}\left(x_i^{(k)} - \tilde{\mu}^{(k)}\right)^2} \tag{26}$$

The iteration will stop if $\left|\tilde{\theta}^{(k)} - \tilde{\theta}^{(k-1)}\right| < \varepsilon$; where ε is a precision parameter.

5. Simulation Study

Through simulation, performance of method of moments proposed by S. M. Shah and M. C. Jaiswal [8] is compared with other competing methods. Other competing methods includes method of maximum likelihood estimator (MLE) proposed by Salvador Pueyo [9] and Quantile filling Algorithm based on the Expectation Maximization Method (QA-EM) by Jun Yang et al. [10]. Performance of each method is measured using bias and mean square error. In the simulation study symmetrical truncated normal and skewed truncated normal distributions are considered to estimate parameters of supplier process distribution.

Bias (B) and Mean Square Error (MSE) is defined as

$$B = E\left(\hat{\theta}\right) - \theta \tag{27}$$

$$MSE = E\left(\hat{\theta} - \theta\right)^2 = Var\left(\hat{\theta}\right) + \left[E\left(\hat{\theta}\right) - \theta\right]^2 \tag{28}$$

Where θ represents the true value of the parameter and $\hat{\theta}$ is its estimate.

One of the factor which affects estimation procedure is proportion of observations truncated or discarded from original process distribution. Here we use α to denote proportion of discarded data which is actually proportion of nonconforming product. For simulation study discarded data proportion is varied from 5% to 15%. While simulating data possible sample sizes are considered as 70, 100, 300, 1000, 5000 and 10,000. From each simulated sample further observations below LSL and above USL is discarded. So while estimating parameters of supplier distribution from truncated simulated sample, effective sample sizes are always less than whatever the considered sample sizes. In case simulated sample size is say 70 and if we discard 5% observations from these then effective sample size of truncated data would be $70 \times (1 - 0.05)$. In the simulation in each scenario 5000 samples is generated using R software.

5.1 Symmetrical Truncated Normal Distribution

Without loss of generality we assume that supplier's process distribution is standard normal. From a simulation point of view, we have considered three symmetric truncated normal distributions with same mean but different variances. First case is explained in detail and on the similar lines other cases are designed.

Case 1: Suppliers process distribution is standard normal and suppose specification limits are LSL=-1.95 and USL=1.95. So in this case using normal distribution properties, proportion of nonconforming product is 5.11%. Suppose supplier has decided to send only conforming product to customer then on the basis of received sample, distribution of process would be truncated normal with

$$Mean = E_T(X) = \mu + \frac{\sigma^2}{M^*}\left(f(a) - f(b)\right)$$

Here $\mu = 0, \sigma^2 = 1$ *and* $a = -1.95, b = 1.95$

= 0+0; Since $f(-1.95) = f(1.95) = 0.05959471$

=0

$$Variance = V_T(X) = \sigma^2\left(\frac{(a-\mu)f(a) - (b-\mu)f(b)}{M^*}\right) + \sigma^2 - \left(\frac{\sigma^2}{M^*}\left(f(a) - f(b)\right)\right)^2$$

$$= 1 \times \left(\frac{(-1.95) \times 0.05959471 - (1.95) * 0.05959471}{F(1.95) - F(-1.95)}\right) + 1;\ since\ f(a) = f(b)$$

= 0.7550448

Simulating this truncated normal distribution having mean=0 and variance= 0.7550448 supplier process distribution parameters are estimated using three different methods. Simulation results are displayed in Table 1.

Table 1. Symmetrical Truncated Normal, $\alpha = 5.11\%$ **true** $\mu = 0$ *and* $\sigma = 1$

[Method*: Moments: Method of moments; MLE: MLE Method by Pueyo S.; QA-EM Algorithm by Jun Yang et al.]

Sample size	Method*	Expected process mean $E(\hat{\mu})$	Bias $E(\hat{\mu}) - \mu$	Mean Square Error of $\hat{\mu}$	Expected process standard deviation	Bias $E(\hat{\sigma}) - \sigma$	Mean Square Error of $\hat{\sigma}$
70	Moments	0.000344	0.000344	0.023002	1.008160	0.008160	0.025970
	MLE	-0.000093	-0.000093	0.022130	1.008153	0.008153	0.024933
	QA-EM	0.000543	0.000543	0.015845	0.952277	-0.047723	0.008461
100	Moments	0.000150	0.000150	0.015698	1.007789	0.007789	0.018392
	MLE	-0.000073	-0.000073	0.014899	1.007277	0.007277	0.017510
	QA-EM	-0.000090	-0.000090	0.012210	0.949790	-0.050210	0.008565
300	Moments	0.000460	0.000460	0.005203	1.002537	0.002537	0.005158
	MLE	0.000558	0.000558	0.004891	1.002468	0.002468	0.004865
	QA-EM	0.000475	0.000475	0.004569	0.971520	-0.028480	0.003808
1000	Moments	-0.000146	-0.000146	0.001494	1.000792	0.000792	0.001429
	MLE	0.000078	0.000078	0.001431	1.000743	0.000743	0.001344
	QA-EM	-0.000105	-0.000105	0.001389	0.987117	-0.012883	0.001209
5000	Moments	0.000190	0.000190	0.000294	1.000003	0.000003	0.000292
	MLE	0.000171	0.000171	0.000276	1.000212	0.000212	0.000276
	QA-EM	0.000090	0.000090	0.000274	0.996047	-0.003953	0.000267
10000	Moments	0.000008	0.000008	0.000146	1.000217	0.000217	0.000146
	MLE	-0.000037	-0.000037	0.000137	1.000488	0.000488	0.000139
	QA-EM	-0.000075	-0.000075	0.000136	0.998015	-0.001985	0.000136

Case 2: Suppose specification limits are LSL=-1.64 and USL=1.64. So in this case proportion of nonconforming product is 10.10%. On the basis of received sample, distribution of process would be

truncated normal with mean=0 and variance= 0.6206959. Simulating this distribution parameters are estimated. Simulation results are displayed in Table 2.

Table 2: Symmetrical Truncated Normal, $\alpha = 10.10\%$ **true** $\mu = 0$ *and* $\sigma = 1$

Sample size	Method*	Expected process mean $E(\hat{\mu})$	Bias $E(\hat{\mu}) - \mu$	Mean Square Error of $\hat{\mu}$	Expected process standard deviation	Bias $E(\hat{\sigma}) - \sigma$	Mean Square Error of $\hat{\sigma}$
70	Moments	0.001834	0.001834	0.046252	1.033445	0.033445	0.071694
	MLE	0.000577	0.000577	0.040594	1.033850	0.033850	0.070934
	QA-EM	0.000605	0.000605	0.019809	0.898952	-0.101048	0.013556
100	Moments	0.001583	0.001583	0.036709	1.028552	0.028552	0.045015
	MLE	0.005087	0.005087	0.190568	1.028798	0.028798	0.048975
	QA-EM	0.000193	0.000193	0.015102	0.914676	-0.085324	0.012637
300	Moments	-0.000053	-0.000053	0.006652	1.006779	0.006779	0.009300
	MLE	-0.000115	-0.000115	0.006344	1.006937	0.006937	0.008936
	QA-EM	-0.000534	-0.000534	0.005346	0.951711	-0.048289	0.005218
1000	Moments	-0.000443	-0.000443	0.001892	1.003209	0.003209	0.002592
	MLE	-0.000446	-0.000446	0.001812	1.003159	0.003159	0.002496
	QA-EM	-0.000892	-0.000892	0.001687	0.977964	-0.022036	0.001968
5000	Moments	0.000191	0.000191	0.000372	1.000102	0.000102	0.000505
	MLE	0.000209	0.000209	0.000357	1.000359	0.000359	0.000489
	QA-EM	0.000013	0.000013	0.000351	0.992638	-0.007362	0.000454
10000	Moments	-0.000003	-0.000003	0.000187	1.000234	0.000234	0.000252
	MLE	-0.000046	-0.000046	0.000177	1.000502	0.000502	0.000245
	QA-EM	-0.000290	-0.000290	0.000175	0.995725	-0.004275	0.000230

Case 3: Assuming specification limits as LSL=-1.43 and USL=1.43, proportion of nonconforming product is 15.27%. On the basis of received conforming product distribution of process would be truncated normal with mean=0 and variance=0.515601. Simulating this distribution suppliers process distribution parameters are estimated. Simulation results are displayed in Table 3.

Table 3: Symmetrical Truncated Normal, $\alpha = 15.27\%$ **true** $\mu = 0$ *and* $\sigma = 1$

Sample size	Method*	Expected process mean $E(\hat{\mu})$	Bias $E(\hat{\mu}) - \mu$	Mean Square Error of $\hat{\mu}$	Expected process standard deviation	Bias $E(\hat{\sigma}) - \sigma$	Mean Square Error of $\hat{\sigma}$
70	Moments	-0.010218	-0.010218	0.432295	1.083579	0.083579	0.266273
	MLE	0.004243	0.004243	0.248861	1.080781	0.080781	0.242071
	QA-EM	-0.001246	-0.001246	0.021403	0.852655	-0.147345	0.020946
100	Moments	0.006143	0.006143	0.716840	1.058630	0.058630	0.245459
	MLE	-0.010611	-0.010611	1.548009	1.066414	0.066414	0.439479
	QA-EM	0.001791	0.001791	0.015694	0.868889	-0.131111	0.016156
300	Moments	-0.000976	-0.000976	0.008765	1.014022	0.014022	0.018585
	MLE	-0.000635	-0.000635	0.008579	1.014625	0.014625	0.018342
	QA-EM	-0.001436	-0.001436	0.006183	0.922670	-0.077330	0.006882
1000	Moments	-0.000744	-0.000744	0.002419	1.003088	0.003088	0.004470
	MLE	-0.000879	-0.000879	0.002333	1.003430	0.003430	0.004355
	QA-EM	-0.002222	-0.002222	0.002017	0.960090	-0.039910	0.002738
5000	Moments	-0.000029	-0.000029	0.000463	1.000534	0.000534	0.000842

	MLE	-0.000043	-0.000043	0.000449	1.000765	0.000765	0.000825
	QA-EM	-0.005636	-0.005636	0.000566	0.977256	-0.022744	0.000726
10000	Moments	0.000117	0.000117	0.000240	1.000090	0.000090	0.000423
	MLE	0.000102	0.000102	0.000232	1.000373	0.000373	0.000413
	QA-EM	0.000373	0.000373	0.000297	0.997040	-0.002960	0.000391

5.2 Skewed Truncated Normal Distribution

Without loss of generality we assume that specification limits are LSL=-1.96 and USL=1.96. Keeping in mind model sampling from skewed truncated normal distributions with specification limits and various values of α (5.00%, 10.11%, and 15.11%) we considered three different supplier's process distributions. In the first case we consider suppliers process distribution is normal with mean=-0.03 and standard deviation=1. In other two cases process distributions are $N(-0.66, 1^2)$, *and* $N(-0.92, 1^2)$ respectively. If nonconforming product is discarded before being sent to customer then for each case we get different truncated normal distribution having means as well as variances are different.

Simulating these truncated normal distributions suppliers process distribution parameters are estimated using three different methods. Simulation results are displayed case wise in Table 4, Table 5 and Table 6 respectively.

Table 4. Skewed Truncated Normal, $\alpha = 5\%$ **true** $\mu = -0.03$ *and* $\sigma = 1$

Sample size	Method*	Expected process mean $E(\hat{\mu})$	Bias $E(\hat{\mu}) - \mu$	Mean Square Error of $\hat{\mu}$	Expected process standard deviation	Bias $E(\hat{\sigma}) - \sigma$	Mean Square Error of $\hat{\sigma}$
70	Moments	-0.035089	-0.005089	0.023620	1.007961	0.007961	0.026440
	MLE	-0.034387	-0.004387	0.022474	1.007388	0.007388	0.025403
	QA-EM	-0.028747	0.001253	0.015976	0.954102	-0.045898	0.008530
100	Moments	-0.029805	0.000195	0.015908	1.009251	0.009251	0.016494
	MLE	-0.030183	-0.000183	0.015205	1.008631	0.008631	0.015692
	QA-EM	-0.027455	0.002545	0.012411	0.952332	-0.047668	0.007802
300	Moments	-0.027950	0.002050	0.005125	1.001717	0.001717	0.005089
	MLE	-0.028280	0.001720	0.004884	1.001722	0.001722	0.004748
	QA-EM	-0.027420	0.002580	0.004567	0.970975	-0.029025	0.003730
1000	Moments	-0.029829	0.000171	0.001515	1.000173	0.000173	0.001434
	MLE	-0.030067	-0.000067	0.001432	1.000541	0.000541	0.001340
	QA-EM	-0.029737	0.000263	0.001387	0.986823	-0.013177	0.001199
5000	Moments	-0.029908	0.000092	0.000303	1.000180	0.000180	0.000287
	MLE	-0.029936	0.000064	0.000283	1.000283	0.000283	0.000275
	QA-EM	-0.029876	0.000124	0.000281	0.995945	-0.004055	0.000264
10000	Moments	-0.030087	-0.000087	0.000148	1.000116	0.000116	0.000145
	MLE	-0.030105	-0.000105	0.000142	1.000322	0.000322	0.000137
	QA-EM	-0.030111	-0.000111	0.000142	0.997622	-0.002378	0.000133

Table 5. Skewed truncated normal, $\alpha = 10.11\%$ **true** $\mu = -0.66$ *and* $\sigma = 1$

Sample size	Method*	Expected process mean $E(\hat{\mu})$	Bias $E(\hat{\mu}) - \mu$	Mean Square Error of $\hat{\mu}$	Expected process standard deviation	Bias $E(\hat{\sigma}) - \sigma$	Mean Square Error of $\hat{\sigma}$
70	Moments	-0.691429	-0.031429	0.246431	1.017714	0.017714	0.042744
	MLE	-0.688082	-0.028082	0.307188	0.996179	-0.003821	0.048294

	QA-EM	-0.607813	0.052187	0.034877	1.010917	0.010917	0.010677
100	Moments	-0.678366	-0.018366	0.032711	1.009662	0.009662	0.022082
	MLE	-0.676324	-0.016324	0.030158	0.997996	-0.002004	0.023098
	QA-EM	-0.617584	0.042416	0.022815	0.992847	-0.007153	0.007913
300	Moments	-0.665831	-0.005831	0.008830	1.001990	0.001990	0.006304
	MLE	-0.665484	-0.005484	0.008380	1.000850	0.000850	0.006011
	QA-EM	-0.638846	0.021154	0.007347	0.970501	-0.029499	0.003373
1000	Moments	-0.662598	-0.002598	0.002493	1.000843	0.000843	0.001770
	MLE	-0.662629	-0.002629	0.002399	1.000928	0.000928	0.001662
	QA-EM	-0.651063	0.008937	0.002268	0.982295	-0.017705	0.001417
5000	Moments	-0.660141	-0.000141	0.000494	0.999936	-0.000064	0.000352
	MLE	-0.660171	-0.000171	0.000477	1.000149	0.000149	0.000333
	QA-EM	-0.656728	0.003272	0.000475	0.994393	-0.005607	0.000317
10000	Moments	-0.660472	-0.000472	0.000256	1.000311	0.000311	0.000178
	MLE	-0.660445	-0.000445	0.000244	1.000495	0.000495	0.000169
	QA-EM	-0.658363	0.001637	0.000243	0.997022	-0.002978	0.000164

Table 6. Skewed Truncated Normal, $\alpha = 15.11\%$ **true** $\mu = -0.92$ *and* $\sigma = 1$

Sample size	Method*	Expected process mean $E(\hat{\mu})$	Bias $E(\hat{\mu}) - \mu$	Mean Square Error of $\hat{\mu}$	Expected process standard deviation	Bias $E(\hat{\sigma}) - \sigma$	Mean Square Error of $\hat{\sigma}$
70	Moments	-0.969967	-0.049967	0.109920	1.017448	0.017448	0.045910
	MLE	-0.905091	0.014909	0.087311	0.933788	-0.066212	0.068177
	QA-EM	-0.878130	0.041870	0.045672	1.050886	0.050886	0.011934
100	Moments	-0.957927	-0.037927	0.062802	1.013221	0.013221	0.028992
	MLE	-0.905520	0.014480	0.053570	0.949977	-0.050023	0.045015
	QA-EM	-0.889118	0.030882	0.033170	1.025767	0.025767	0.009618
300	Moments	-0.928714	-0.008714	0.014331	1.001384	0.001384	0.007704
	MLE	-0.903748	0.016252	0.015218	0.978111	-0.021889	0.012278
	QA-EM	-0.894757	0.025243	0.011354	0.984694	-0.015306	0.004104
1000	Moments	-0.922018	-0.002018	0.003861	1.000472	0.000472	0.002172
	MLE	-0.916056	0.003944	0.004368	0.996354	-0.003646	0.002778
	QA-EM	-0.903478	0.016522	0.003537	0.980299	-0.019701	0.001642
5000	Moments	-0.920434	-0.000434	0.000766	1.000174	0.000174	0.000440
	MLE	-0.920086	-0.000086	0.000741	1.000022	0.000022	0.000411
	QA-EM	-0.914645	0.005355	0.000737	0.993750	-0.006250	0.000389
10000	Moments	-0.920307	-0.000307	0.000371	1.000308	0.000308	0.000214
	MLE	-0.920089	-0.000089	0.000357	1.000304	0.000304	0.000202
	QA-EM	-0.916813	0.003187	0.000359	0.996516	-0.003484	0.000195

5.3 Results

1. **Symmetrical Truncated Normal Distribution** (refer Table 1-3)

In case of symmetrical truncated normal distribution we have set true value of $\mu = 0$ *and* $\sigma = 1$. It is observed that as the sample size increases, the estimates of μ as well as σ of the three methods regardless of magnitude of α converge to the corresponding true values.

Considering all sample sizes on an average when $\alpha = 5.11\%$, μ estimated by MLE method is more close to the true value as compared to other two methods, however μ estimated by other two methods is also accurate. On an average when $\alpha = 10.10\%$, μ estimated by QA-EM is more close to the true value as compared to other two methods, but other two methods also accurately estimated μ. When $\alpha = 15.27\%$, on an average μ estimated by method of moments is more close to the true value as compared to other two methods, though other two methods also accurately estimated μ. Regarding precision of estimate of μ on the basis of Mean Square Error it is found that QA-EM method is superior as compared to other two methods. But when we compare MLE and method of moments regarding precision of estimate of μ, then method moments is best as compare to MLE.

Considering all sample sizes on an average for all values of α, σ estimated by method of moments as well as MLE is accurate but σ estimated by QA-EM is not accurate and it under estimates σ. Regarding precision of estimate of σ on the basis of Mean Square Error it is found that QA-EM method is superior as compared to other two methods. But when we compare MLE and method of moments regarding precision of estimate of σ, then method moments is best as compared to MLE.

2. Skewed Truncated Normal Distribution (refer Table 4-6)

In case of skewed truncated normal distribution we have set true value of $\mu = -0.03$ in case one, $\mu = -0.66$ in case two and $\mu = -0.92$ in case three. In all cases true value of σ is 1. It is observed that as the sample size increases, the estimates of μ as well as σ of the three methods regardless of magnitude of α converge to the corresponding true values.

Considering all sample sizes on an average when $\alpha = 5.11\%$, μ estimated by method of moments and MLE method is more close to the true value as compared to QA-EM method. This observation is same for other values of α. Regarding precision of estimate of μ, QA-EM method is superior as compare to other methods.

Considering all sample sizes on an average for all α, σ estimated by method of moments is more close to the true value as compared to other methods. When $\alpha = 5\%$ *and* 10.11%, σ estimated by QA-EM is not accurate. For $\alpha = 15.11\%$, σ estimated by MLE methods is not accurate. Regarding precision of estimate of σ, QA-EM is better.

In summary simulation indicates that method of moments is an effective in general to estimate parameters of normal distribution in all cases as compared to other two methods.

6. Basic PCIs and Probability based PCIs

6.1 Basic Univariate PCIs

Basic PCIs C_p, C_{pk} *and* C_{pm} are defined under the assumptions that

a) the process is in statistical control

b) a tolerance region of a quality characteristic is specified by lower and upper specification limits (LSL and USL) and a target value T which is the midpoint of the specification limits and

c) the process measurement (X) are normally distributed with mean μ and variance σ^2.

The C_p index is defined as

$$C_p = \frac{USL - LSL}{6\sigma} \tag{29}$$

C_p measures only the potential capability of the process. Due to the inability of C_p to consider process target, several indices have been proposed that attempt to take the target value T into account which include C_{pk} and C_{pm}. The index C_{pk} is defined as

$$C_{pk} = \min(C_{pu}, C_{pl}) \,, \tag{30}$$

where C_{pu} and C_{pl} are unilateral measures of process capability also known as upper and lower capability indices and they are defined as follows

$$C_{pu} = \frac{USL - \mu}{3\sigma} \tag{31}$$

$$C_{pl} = \frac{\mu - LSL}{3\sigma} \tag{32}$$

Third basic univariate PCI is given by

$$C_{pm} = \frac{USL - LSL}{6\sigma_T}, \ where\ \sigma_T = \sqrt{E(X-T)^2} \tag{33}$$

In practice parameters μ and σ are replaced by their estimators $\hat{\mu}$ and $\hat{\sigma}$ computed from sample $X_1, X_2, ..., X_n$. That is

$$\hat{\mu} = \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad \hat{\sigma} = S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2}\ .$$

Estimator for σ_T is $\hat{\sigma}_T = \sqrt{\hat{\sigma}^2 + \left(\overline{X} - T\right)^2}$.

C_{pm} (also called Taguchi index) is more aligned with Taguchi loss function because it gives more importance to target. Magnitude of the difference between C_p and the other two indices reflects improvement that can be realized by moving the process mean to the target, which is often easier than reducing variation. Desirable value of each PCI is greater than or equal to 1.

6.2 Univariate Probability based PCIs

Khadse and Shinde [11] introduced probabilities p_1, p_2 *and* p_3 to construct alternative forms of C_p, C_{pk} *and* C_{pm} under the assumptions, as mentioned in the beginning of 6.1. Defined probabilities are

$$p_1 = P(LSL \le X \le USL \ |\mu = T, \sigma^2) \tag{34}$$

$$p_2 = P(LSL \le X \le USL \ |\ \mu, \sigma^2) \tag{35}$$

$$p_3 = P(LSL \le X \le USL \ |\mu = T, \sigma^2 = \sigma_T^2) \tag{36}$$

Alternative definitions for C_p, C_{pk} *and* C_{pm} using p_1, p_2 *and* p_3 are given below.

$$C_{p(p1)} = -\frac{1}{3}\Phi^{-1}\left(\frac{1-p_1}{2}\right) \tag{37}$$

$$C_{pk(p2)} = -\frac{1}{3}\Phi^{-1}\left(\frac{1-p_2}{2}\right) \tag{38}$$

$$C_{pm(p3)} = -\frac{1}{3}\Phi^{-1}\left(\frac{1-p_3}{2}\right) \tag{39}$$

where Φ^{-1} denotes the inverse distribution function of the standard normal distribution. They have estimated $C_{p(p1)}$, $C_{pk(p2)}$ *and* $C_{pm(p3)}$ using $\hat{p}_1, \hat{p}_2$ *and* $\hat{p}_3$ respectively. Khadse and Shinde [11] mathematically proved that under the assumptions of normality and symmetric tolerance the following equivalences hold.

(i) $C_p = C_{p(p1)}$,

(ii) $C_{pm} = C_{pm(p3)}$.

They have noted that C_{pk} and $C_{pk(p2)}$ are not mathematically equivalent though they suggested $C_{pk(p2)}$ as an alternative to C_{pk} as it is directly based on fraction conforming probability. The importance of probability based indices is that they are easy to extend in multivariate setup.

Illustrative Example: This study involved a manufacturer and supplier of a brake system. In brake system of vehicle, Master Cylinder is one of the important components. Master Cylinder converts force input given by the driver into hydraulic pressure. Whole brake system is affected if there is any problem with Master Cylinder. There are two types of Master Cylinders first one is Single Master Cylinder (it has only one chamber) and second one is Tandem Master Cylinder (it has two separate chambers). Port hole diameter is one of the critical characteristic of the Tandem Master Cylinder. The USL, LSL and target value for port hole diameter were 14.3mm, 14.1mm and 14.2mm respectively.

Suppose Vehicle Manufacturer Company received batch of 1000 Tandem Master Cylinder (TMC) from brake system supplier as a sample. In order to check supplier's process capability regarding port hole diameter of TMC, port hole diameter is measured of each TMC using snap gauge. Here we give generated observations in the form of histogram in Figure 1.

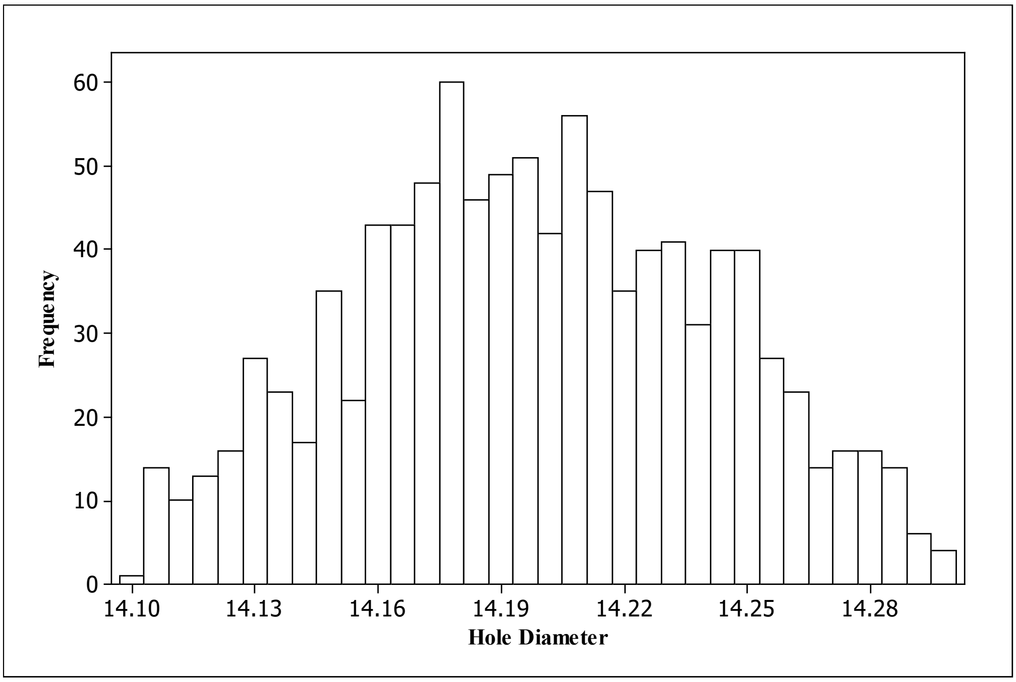

Figure 1. Histogram of Supplier's Process Distribution Based on Received Sample

Using histogram it is found that supplier's process distribution is normal and received products process distribution is doubly truncated normal with a=14.1, b=14.3 and unknown μ *and* σ which to be estimated to measure supplier's process capability.

Applying method of moments proposed by Shah and M. C. Jaiswal [8] to truncated data estimated values of μ *and* σ parameters of normal distribution are 14.1984 and 0.0502.

Using alternative definitions of C_p, C_{pk} *and* C_{pm} based on p_1, p_2 *and* p_3

$$p_1 = P(LSL \le X \le USL \,|\, \mu = T, \sigma^2);\; X \sim N(\mu = 14.2, \sigma^2 = 0.0502^2)$$

$$= P(14.1 \le X \le 14.3) = 0.953632$$

$$p_2 = P(LSL \le X \le USL \,|\, \mu, \sigma^2);\; X \sim N(\mu = 14.1984; \sigma^2 = 0.0502^2)$$

$$= P(14.1 \le X \le 14.3) = 0.953521$$

$$p_3 = P(LSL \le X \le USL \,|\, \mu = T, \sigma^2 = \sigma_T^2);\; X \sim N(\mu = 14.2, \sigma_T^2 = 0.050225^2)$$

$$= P(14.1 \le X \le 14.3) = 0.953523$$

Estimated values of $C_{p(p1)}, C_{pk(p2)}$ *and* $C_{pm(p3)}$ using p_1, p_2 *and* p_3 (refer equation (37-39)) are 0.6640, 0.6636 and 0.6636. Supplier's process is neither potentially capable nor actually capable. To improve process capability, variation in the process must be reduced. As the estimated value of $C_{pm(p3)}$ is just less than estimated value of $C_{p(p1)}$, the process is just shifted from target.

7. Conclusions

Most of the time received sample of products from supplier is hundred percent conforming to the specification due to products categorized as conforming and nonconforming by supplier before being sent to customer. If we assume supplier's process distribution is normal then the process distribution identified on the basis of received sample of products is truncated normal. From a customer's point of view, to measure supplier's process capability there is a need of estimating parameters of process distribution using truncated normal data. Initially we discussed theoretical background of all cases of truncated normal distribution. To deal with estimation of parameters of truncated normal distribution we have discussed method of moments, method of maximum likelihood estimations, method of maximum likelihood starting from re-parameterization proposed by Salvador Pueyo [9] and Quantile-filling Algorithm (QA) based on EM algorithm proposed by Jun Yang et al. [10]. While comparing these methods using simulation, we have not considered maximum likelihood estimation method because it is already compared by Shah and Jaiswal [8] with method of moments and they have preferred method of moments due to MLE being complicated and laborious. Using simulation we conclude that though method of moments is traditional but its performance is better with respect to accuracy of estimates as well as more or less with respect to precision as compared to other two methods. Method of maximum likelihood starting from re-parameterization and Quantile-filling Algorithm (QA) based on EM algorithm, these two methods are iterative in nature so they carry drawbacks of iterative methods and within these two methods method of maximum likelihood starting from re-parameterization is better. Quantile-filling Algorithm (QA) based on EM is complicated and laborious and did not estimate standard deviation accurately though precision of estimated parameters is the best. At the end, we have discussed basic PCIs and their alternatives based on conforming type probabilities, through an illustrative example. Another interesting topic for the future is measurement of supplier's multivariate process capability. This can be studied with suitable parameter estimation method by extending truncated normal into multivariate truncated normal and by extending basic probability based PCIs into multivariate PCIs.

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