New Laplace Variational Iterative Technique for analytical solution of Two Dimensional Heat Equations

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Abstract: In this paper,a new semi analytical technique derived from the combination of Laplace Transform method and Variational iterative method is employed to find the solution of two dimensional heat equations. Numerical results are discussed to demonstrate the efficiency and accuracy of the proposed method.

Keywords: *Laplace transform, Variational iterative method, Two-dimensional Heat equation, Numerical examples.*

I. INTRODUCTION

The heat equation is an important second order parabolic partial differential equation.The theory of heat equations was first introduced by Joseph Fourier in 1822. It arises in various applications of engineering and sciences in which we have to model how a quantity such as heat transfers through a specified region.The two dimensional heat equation is described as the following

$$\frac{\partial u(x,y,t)}{\partial t} = c^2\nabla^2 u(x,y,t)$$

Laplace transform is a an important analytical method for solving the differential equation by interpreting the differential equations into simple equations and convolutions into multiplications. Laplace transform method can be applied to solve a number of applications arising in engineering and sciences. Variational iterative method is also a well-known numerical method for solving the differential equations.The exact solution of the differential equations can be obtained ,if exists, by using variational iterations.

A number of mathematical methods have been introduced for solving two dimensional heat equations. Finite difference method has been used for solving two-dimensional heat equations in [1]. Chebyshev series solution of the two dimensional heat equations has been introduced in [2]. The combination of Finite Difference Method and Collocation method has been developed for solving two-dimensional heat equations in [3]. Radial basis function method has been demonstrated to solve two dimension heat equations in [4]. Variational iteration technique has been introduced in [5], for solving nonlinear equations.

II. LAPLACE TRANSFORM METHOD:

Let $u(t)$ be a function of t defined for all positive values of t. Then the Laplace transforms of $u(t)$, represented as $L\{u(t)\}$ and is defined as:

$$L\{u(t)\} = \int_0^\infty e^{-pt}u(t)dt = \bar{u}(p),$$

provided the integral exists and $'p'$ is a parameter which may be a real or complex number. Therefore

$$L\{u(t)\} = \bar{u}(p),$$

that is

$$u(t) = L^{-1}\{\bar{u}(p)\}.$$

The term $L^{-1}\{\bar{u}(p)\}$, is called the inverse Laplace transform of $\bar{u}(p)$.

III. Linearity Property of Laplace transform Method:

Let $v(t),\ w(t)$ be two functions of t defined for all positive values of t. Then

$$L\{a.v(t) + b.w(t)\} = a.L\{v(t)\} + b.L\{w(t)\}$$

where a and b are arbitrary constants.

IV. Laplace transform for differentiation:

Let $u(t)$ be a function of t defined for all positive values of t. Then, the Laplace transform of n^{th} derivative of function $u(t)$ is

$$L\left[\frac{d^n(u(t))}{dt^n}\right] = p^n\bar{u}(p) - p^{n-1}u(0) - p^{n-2}u'(0) - p^{n-3}u''^{(0)} - \cdots - pu^{(n-2)}(0) - u^{(n-1)}(0)$$

where $\bar{u}(p) = L\{u(t)\}$.

V. Linearity Property of Inverse Laplace transform:

Let $v(t),\ w(t)$ be two functions of t defined for all positive values of t. Let $\bar{v}(p)$ and $\bar{w}(p)$ be the functions of s such that $\bar{v}(p) = L\{v(t)\}$ and $\bar{w}(p) = L\{w(t)\}$.

Then

$$L^{-1}\{c.\bar{v}(p) + d.\bar{w}(p)\} = c.L^{-1}\{\bar{v}(p)\} + d.L\{\bar{w}(p)\} = c.v(t) + d.w(t)$$

where c and d are arbitrary constants.

VI. Variational iterative method (VIM)

Variational iteration method is an important method used to solve many problems arising in various applications of engineering and Sciences. The nonlinear terms can be handled with the help of variational iteration method. Consider the differential equations,

$$\boldsymbol{l}u(x,y,t) + \boldsymbol{n}u(x,y,t) = \boldsymbol{g}(x,y,t) \tag{1}$$

with the initial conditions

$$u(x,y,0) = \boldsymbol{h}(x,y) \tag{2}$$

where $\boldsymbol{l}$ is a linear operator of the first order, $\boldsymbol{n}$ is nonlinear operator and $\boldsymbol{g}$ is a nonhomogeneous term. From variational iteration method, construct a correction functional as

$$u_{m+1} = u_m + \int_0^t \lambda[\boldsymbol{l}u_m(x,y,p) + \boldsymbol{n}\tilde{u}_m(x,y,p) - \boldsymbol{g}(x,y,p)]dp \tag{3}$$

where λ is a known as Lagrange's multiplier and $\boldsymbol{m}$ denotes the m^{th} approximations, $\tilde{u}_m$ is restricted function, i.e. $\delta\tilde{u}_m = 0$. The successive approximation u_{m+1} of the solution u will be obtained by using λ and u_0. The solution is

$$u = \lim_{m\to\infty} u_m$$

VII. New Semi Analytical Method for Solving two dimensional Heat Equations

The new semi analytical method is based on the combination of Laplace transform and variational iterative method used to solve the various problems of partial differential equations. The process for solving the partial differential equations by using this semi analytical is presented in this section as given below.

Assume that $\boldsymbol{l}$ is an operator of the first order $\frac{\partial}{\partial t}$. Equation (1) becomes

$$\frac{\partial}{\partial t}u(x,y,t) + \boldsymbol{n}u(x,y,t) = \boldsymbol{g}(x,y,t) \tag{4}$$

Taking Laplace transform on both sides of (4), we obtain

$$L\left\{\frac{\partial}{\partial t}u(x,y,t)\right\} + L\{\boldsymbol{n}u(x,y,t)\} = L\{\boldsymbol{g}(x,y,t)\} \tag{5}$$

$$pL\{u(x,y,t)\} - \boldsymbol{h}(x,y) = L\{\boldsymbol{g}(x,y,t)\} - L\{\boldsymbol{n}u(x,y,t)\} \tag{6}$$

Applying inverse Laplace transform on both sides of (6), we obtain

$$u(x,y,t) = \boldsymbol{G}(x,y,t) - L^{-1}\left[\frac{1}{p}L\{\boldsymbol{n}u(x,y,t)\}\right] \tag{7}$$

where $\boldsymbol{G}$ is the term arising from source term and given initial condition. From the correctional functional of the variational iteration method

$$u_{m+1}(x,y,t) = \boldsymbol{G}(x,y,t) - L^{-1}\left[\frac{1}{p}L\{\boldsymbol{n}u_m(x,y,t)\}\right] \tag{8}$$

Equation (8) represents the new modified correction functional of Laplace transform of variational iteration method, the solution is given by

$$u(x,y,t) = \lim_{m\to\infty} u_m(x,y,t)$$

VIII. Numerical Examples

In order to illustrate the efficiency of the proposed semi analytical technique,examples are given in this section.

Example 1: Consider the following two dimensional heat equation

$$\frac{\partial u(x,y,t)}{\partial u} = \nabla^2 u(x,y,t) \tag{9}$$

with initial conditions

$$u(\mathrm{x},\mathrm{y},0) = \mathrm{sinx\ cosy}$$

Applying Laplace transform on both sides of (9), we obtain

$$L\left\{\frac{\partial u(x,y,t)}{\partial u}\right\} = L\{\nabla^2 u(x,y,t)\} \tag{10}$$

This implies

$$pL\{u(x,y,t)\} - u(x,y,0) = L\{\nabla^2 u(x,y,t)\}$$

Applying initial conditions, we obtain

$$pL\{u(x,y,t)\} = sinx\ cosy + L\{\nabla^2 u(x,y,t)\}$$

Divide both sides by p, we obtain

$$L\{u(x,y,t)\} = \frac{sinx\ cosy}{p} + \frac{1}{p}L\{\nabla^2 u(x,y,t)\} \tag{11}$$

Applying inverse Laplace transform on both sides of (11), we obtain

$$u = sinx\ cosy + L^{-1}\left[\frac{1}{p}L\{\nabla^2 u(x,y,t)\}\right] \tag{12}$$

Using iteration method, from (12), we obtain

$$u_{m+1} = sinx\ cosy + L^{-1}\left[\frac{1}{p}L\{\nabla^2 u_m\}\right] \tag{13}$$

From (13), we obtain

$$u_0 = sinx\ cosy$$

$$u_1 = sinx\ cosy(1-2t)$$

$$u_2 = sinx\ cosy\left(1 - 2t + \frac{(2t)^2}{2!}\right)$$

$$u_3 = sinx\ cosy\left(1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!}\right)$$

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$$u_m = sinx\ cosy\left(1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \cdots + \frac{(-1)^m(2t)^m}{m!}\right)$$

The solution is

$$u = \lim_{n\to\infty} u_m$$

After simplification, we obtain

$$u = sinx\ cosy\left(1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} \dots\right)$$

$$u = \text{sinx cosy}\ e^{-2t} \tag{14}$$

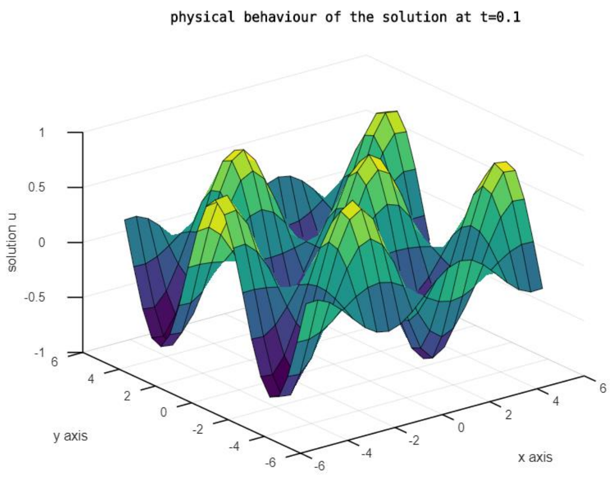

Figure 1

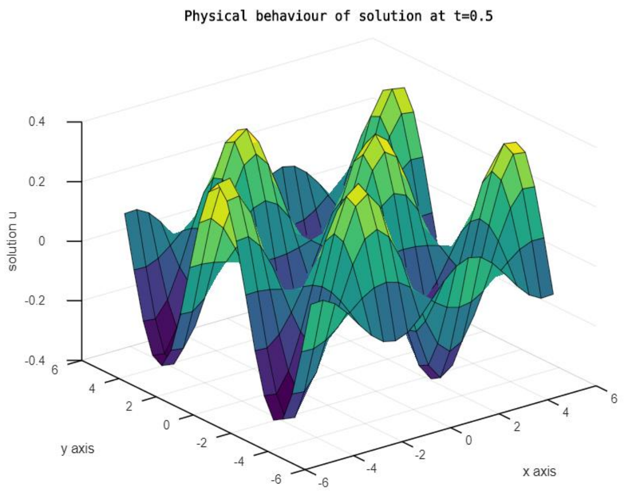

Figure 2

Figure 1 & Figure 2 represent the Physical behaviour of the solution of Example 1 at t = 0.1 and t = 0.5 respectively.

Example 2: Consider the two-dimensional Heat equation

$$\frac{\partial u(x,y,t)}{\partial t} = \nabla^2 u(x,y,t) \tag{15}$$

where

$$u(x,y,0) = e^{x+y}$$

Applying Laplace transform on both sides of (15), we obtain

$$L\left\{\frac{\partial u(x,y,t)}{\partial t}\right\} = \mathrm{L}\{\nabla^2 u(x,y,t)\}$$

This implies

$$pL\{u(x,y,t)\} - u(x,y,0) = L\{\nabla^2 u(x,y,t)\}$$

Applying initial conditions, we obtain

$$pL\{u(x,y,t)\} = e^{x+y} + L\{\nabla^2 u(x,y,t)\}$$

Divide by p, we obtain

$$L\{u(x,y,t)\} = \frac{e^{x+y}}{p} + \frac{1}{p} L\{\nabla^2 u(x,y,t)\} \tag{16}$$

Applying inverse Laplace transform on both sides of (16), we obtain

$$u = e^{x+y} + L^{-1}\left[\frac{1}{p} L\{\nabla^2 u(x,y,t)\}\right] \tag{17}$$

Using iteration method, from (17), we obtain

$$u_{m+1} = e^{x+y} + L^{-1}\left[\frac{1}{p} L\{\nabla^2 u_m\}\right] \tag{18}$$

From (18), we obtain

$$u_0 = e^{x+y}$$

$$u_1 = e^{x+y}(1+2t)$$

$$u_2 = e^{x+y}\left(1+2t+\frac{(2t)^2}{2!}\right)$$

$$u_3 = e^{x+y}\left(1+2t+\frac{(2t)^2}{2!}+\frac{(2t)^3}{3!}\right)$$

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$$u_m = e^{x+y}\left(1+2t+\frac{(2t)^2}{2!}+\frac{(2t)^3}{3!}+\cdots+\frac{(2t)^m}{m!}\right)$$

The solution is obtained as

$$u = \lim_{n\to\infty} u_m$$

$$u = e^{x+y}\left(1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \cdots\right)$$

$$u = e^{x+y}(e^{2t})$$

$$u = e^{x+y+2t}$$

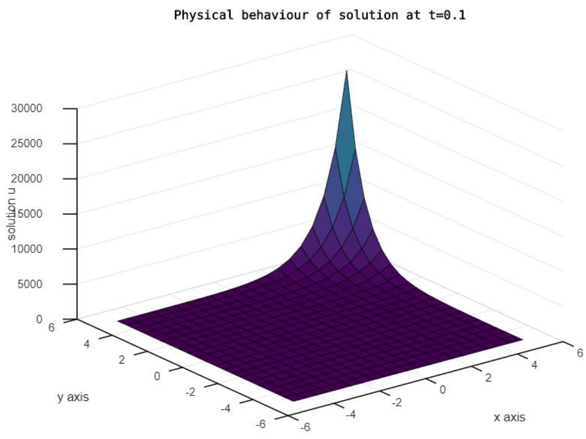

Figure 3

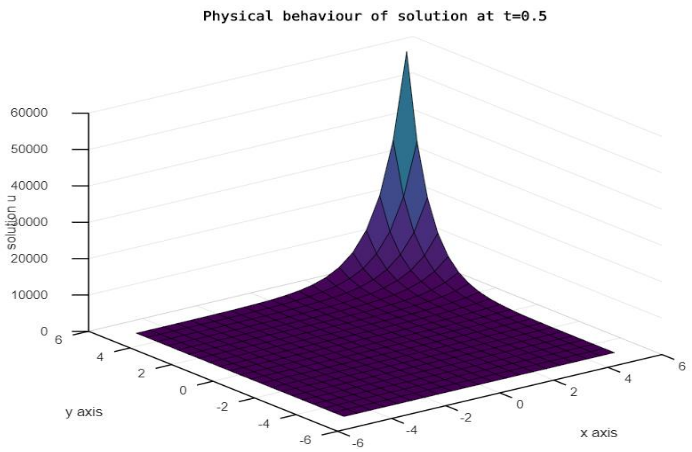

Figure 4

Figure 3 & Figure 4, represent the Physical behaviour of the solution of Example 2 at t = 0.1 and t = 0.5 respectively.

Example 3: Consider the two-dimensional Heat equation

$$\frac{\partial u(x,y,t)}{\partial t} = \nabla^2 u(x,y,t) \tag{19}$$

where

$$u(x,y,0) = (1-y)e^x$$

Applying Laplace transform on both sides of (19), we obtain

$$L\left\{\frac{\partial u(x,y,t)}{\partial t}\right\} = \mathrm{L}\{\nabla^2 u(x,y,t)\}$$

This implies

$$pL\{u(x,y,t)\} - u(x,y,0) = L\{\nabla^2 u(x,y,t)\}$$

Applying initial conditions, we obtain

$$pL\{u(x,y,t)\} = (1-y)e^{x} + L\{\nabla^2 u(x,y,t)\}$$

Divide by p, we obtain

$$L\{u(x,y,t)\} = \frac{(1-y)e^{x}}{p} + \frac{1}{p}L\{\nabla^2 u(x,y,t)\} \tag{20}$$

Applying inverse Laplace transform on both sides of (20), we obtain

$$u = (1-y)e^{x} + L^{-1}\left[\frac{1}{p}L\{\nabla^2 u(x,y,t)\}\right] \tag{21}$$

Using iteration method, from (21), we obtain

$$u_{m+1} = (1-y)e^{x} + L^{-1}\left[\frac{1}{p}L\{\nabla^2 u_m\}\right] \tag{22}$$

From (22), we obtain

$$u_0 = (1-y)e^{x}$$

$$u_1 = (1-y)e^{x}(1+t)$$

$$u_2 = (1-y)e^{x}\left(1+t+\frac{(t)^2}{2!}\right)$$

$$u_3 = (1-y)e^{x}\left(1+t+\frac{(t)^2}{2!}+\frac{(t)^3}{3!}\right)$$

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$$u_m = (1-y)e^{x}\left(1+t+\frac{(t)^2}{2!}+\frac{(t)^3}{3!}+\cdots+\frac{(t)^m}{m!}\right)$$

The solution is obtained as

$$u = \lim_{n\to\infty} u_m$$

$$u = (1-y)e^{x}\left(1+t+\frac{(t)^2}{2!}+\frac{(t)^3}{3!}+\cdots\right)$$

$$u = (1-y)e^{x}(e^{t})$$

$$u = (1-y)e^{x+t}$$

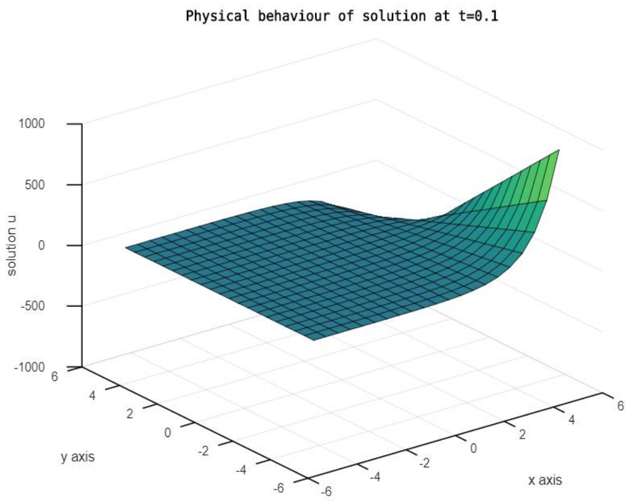

Figure 5

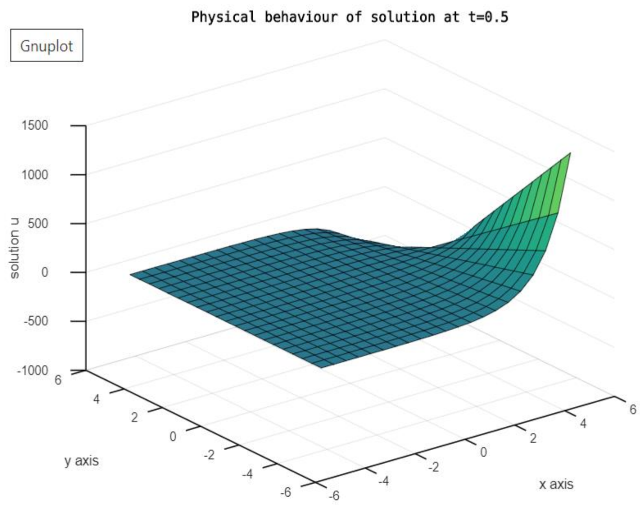

Figure 6

Figure 5 & Figure 6, represent the Physical behaviour of the solution of Example 3 at t = 0.1 and t = 0.5 respectively.

Example 4: Consider the two-dimensional Heat equation

$$\frac{\partial u(x,y,t)}{\partial t} = \nabla^2 u(x,y,t) \tag{23}$$

where

$$u(x,y,0) = sinhx\ sinhy$$

Applying Laplace transform on both sides of (23), we obtain

$$L\left\{\frac{\partial u(x,y,t)}{\partial t}\right\} = \mathrm{L}\{\nabla^2 u(x,y,t)\}$$

This implies

$$pL\{u(x,y,t)\} - u(x,y,0) = L\{\nabla^2 u(x,y,t)\}$$

Applying initial conditions, we obtain

$$pL\{u(x,y,t)\} = sinhx\ sinhy + L\{\nabla^2 u(x,y,t)\}$$

Divide by p, we obtain

$$L\{u(x,y,t)\} = \frac{sinhx\ sinhy}{p} + \frac{1}{p}L\{\nabla^2 u(x,y,t)\} \tag{24}$$

Applying inverse Laplace transform on both sides of (24), we obtain

$$u = sinhx\ sinhy + L^{-1}\left[\frac{1}{p}L\{\nabla^2 u(x,y,t)\}\right] \tag{25}$$

Using iteration method, from (25), we obtain

$$u_{m+1} = sinhx\ sinhy + L^{-1}\left[\frac{1}{p}L\{\nabla^2 u_m\}\right] \tag{26}$$

From (26), we obtain

$$u_0 = sinhx\ sinhy$$

$$u_1 = sinhx\ sinhy(1 + 2t)$$

$$u_2 = sinhx\ sinhy\left(1 + 2t + \frac{(2t)^2}{2!}\right)$$

$$u_3 = sinhx\ sinhy\left(1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!}\right)$$

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$$u_m = sinhx\ sinhy\left(1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \cdots + \frac{(2t)^m}{m!}\right)$$

The solution is obtained as

$$u = \lim_{n\to\infty} u_m$$

$$u = sinhx\ sinhy\left(1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \cdots\right)$$

$$u = sinhx\ sinhy(e^{2t})$$

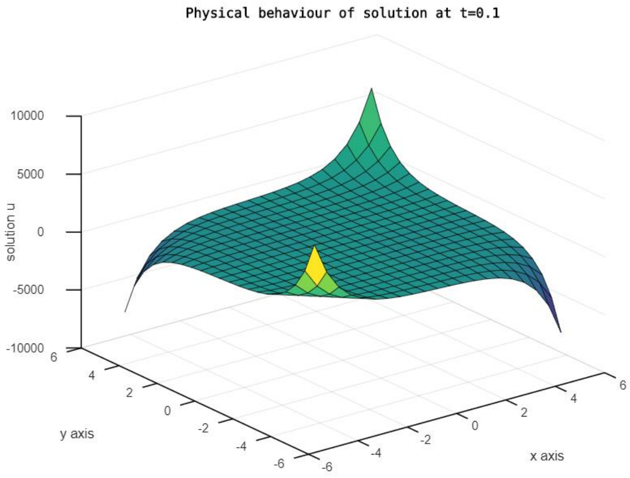

Figure 7

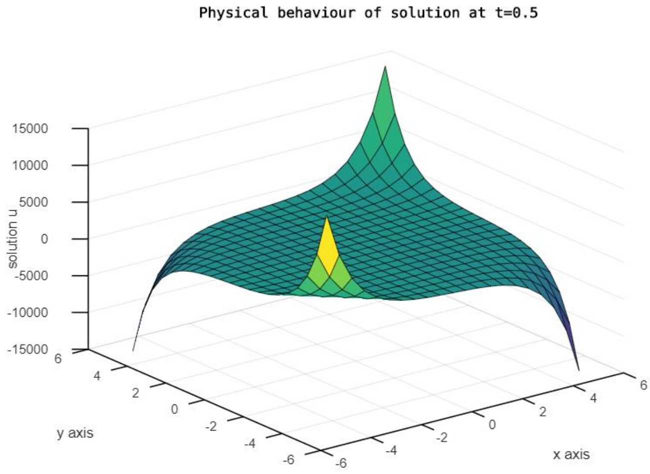

Figure 8

Figure 7 & Figure 8, represent the Physical behaviour of the solution of Example 4 at t = 0.1 and t = 0.5 respectively.

Example 5: Consider the two-dimensional Heat equation

$$\frac{\partial u(x,y,t)}{\partial t} = \nabla^2 u(x,y,t) \tag{27}$$

where

$$u(x,y,0) = e^y - xe^y$$

Applying Laplace transform on both sides of (27), we obtain

$$L\left\{\frac{\partial u(x,y,t)}{\partial t}\right\} = \mathrm{L}\{\nabla^2 u(x,y,t)\}$$

This implies

$$pL\{u(x,y,t)\} - u(x,y,0) = L\{\nabla^2 u(x,y,t)\}$$

Applying initial conditions, we obtain

$$pL\{u(x,y,t)\} = e^y - xe^y + L\{\nabla^2 u(x,y,t)\}$$

Divide by p, we obtain

$$L\{u(x,y,t)\} = \frac{e^y - xe^y}{p} + \frac{1}{p}L\{\nabla^2 u(x,y,t)\} \tag{28}$$

Applying inverse Laplace transform on both sides of (28), we obtain

$$u = e^y - xe^y + L^{-1}\left[\frac{1}{p}L\{\nabla^2 u(x,y,t)\}\right] \quad (29)$$

Using iteration method, from (29), we obtain

$$u_{m+1} = e^y - xe^y + L^{-1}\left[\frac{1}{p}L\{\nabla^2 u_m\}\right] \quad (30)$$

From (30), we obtain

$$u_0 = e^y - xe^y$$

$$u_1 = e^y - xe^y(1+t)$$

$$u_2 = e^y - xe^y\left(1 + t + \frac{(t)^2}{2!}\right)$$

$$u_3 = e^y - xe^y\left(1 + t + \frac{(t)^2}{2!} + \frac{(t)^3}{3!}\right)$$

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$$u_m = e^y - xe^y\left(1 + t + \frac{(t)^2}{2!} + \frac{(t)^3}{3!} + \cdots + \frac{(t)^m}{m!}\right)$$

The solution is obtained as

$$u = \lim_{n\to\infty} u_m$$

$$u = e^y - xe^y\left(1 + t + \frac{(t)^2}{2!} + \frac{(t)^3}{3!} + \cdots\right) = (e^y - xe^y)(e^t)$$

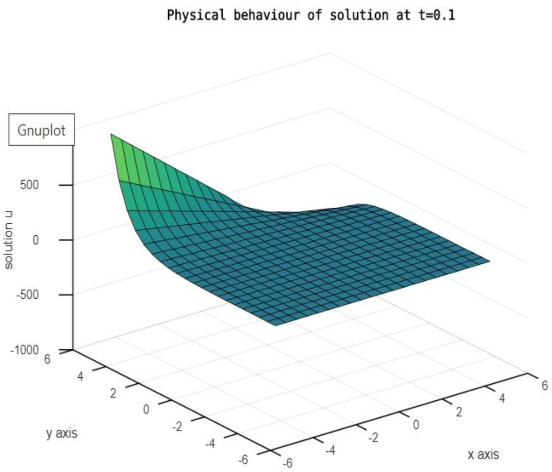

Figure 9

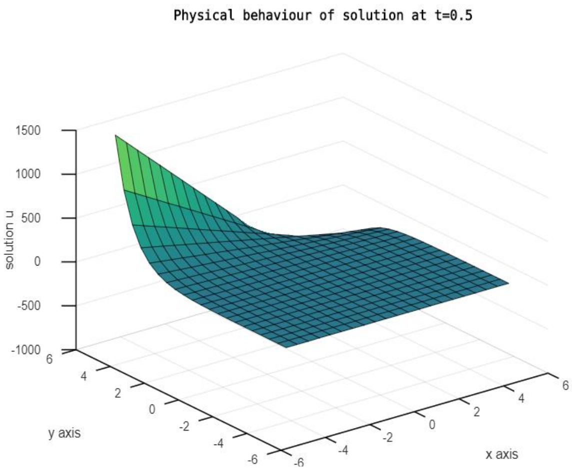

Figure 10

Figure 9 & Figure 10 , represent the Physical behaviour of the solution of Example 5 at t = 0.1 and t = 0.5 respectively.

CONCLUSION

From the solved numerical examples, It is observed that the new semi analytical technique which is combination of Laplace transform and modified variational iterative method is an efficient mathematical method to solve two-dimensional heat equations in simple steps. The proposed Mathematical method may be used for solving two dimensional and three-dimensional linear and non-linear heat equations in future.

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