Comparative Study of Iterative Methods for Solving Non-Linear Equations

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Abstract:

In numerical analysis, methods for finding root play a pivotal role in the field of many real and practical applications. The efficiency of numerical methods depends upon the convergence rate (how fast the particular method converges). The objective of this study is to compare Bisection method, Newton-Raphson method and False Position Method with their limitations and also analyse them to know which of them is more preferred. Limitations of these methods have allowed presenting the latest research in the area of iterative processes for solving non-linear equations. This paper analysis the field of iterative methods which are developed in recent years with its future scope.

Introduction

The beginning of iterative technique is from approximate values and these values are used in repetitive formula to give additional approximate values by continuously trying the repetitive formula and accurate solution is obtained. Iterative technique makes us able to find a root to any particular degree of correctness.

An approach to find initial values to the root of f(x) = 0 is construction of the values of the function and afterwards use intermediate value theorem which states:

"If f is a continuous function on interval [a,b] and f(a) and f(b) have opposite signs then there exists at least one real root of f(x) = 0 in the interval (a,b)".

After getting the interval containing the root, take some point in the interval as the initial approximations to the root. [1]

Order of convergence of iterative methods:

The convergence fastness is judged by its order of convergence. Higher the order of convergence means that error in the successive approximations obtained decreases more rapidly. The order of convergence is as follows:

An iterative technique is called order of p if p of the largest real number (≥ 1) such that

$|e_{k+1}| \leq A|e_k|^p$ (k→∞)

For some constant A≠0, where $e_k = x_k - £$ and $e_{k+1} = x_{k+1} - £$ are the errors in kth and (k+1)th approximation respectively.

The constant A is called asymptomatic error constant.

It follows from the above definition [1] that number of significant digits in each approximation increases p times than that in previous approximation. It should be noted that for linearly convergent methods $|e_{k+1}| \leq A|e_k|$ so smaller the value of A means that error in the next approximation decreases more rapidly. Smaller the value of magnitude of 'A' means higher the speed the convergence [2]. The usual root finding methods include Bisection, False Position method and Newton Raphson method and many more diverse methods which converges to the root at unlike rates.[2].

Literature Review

- **Beong In Yun (2012)** said that iterative values are unchangeable as a result values converge to the root without the concern of initial values. Some techniques based on the numerical integration processes were planned by him.[3]
- **M.A. Hafiz, Salwa M.H., Al-Goria (2013)** proposed advanced order iterative techniques excluded from second derivative for attaining the solution. They considered the efficiency index and computational order of the novel techniques.[4]
- **Ehiwario, J.C., Aghamie, S.O. (2014)** suggested that the convergence of Bisection is sure but its rate of convergence is very unrushed. Thus it is rather difficult to use for systems of equations.[5]
- **Saba Akram, Qurrat Ul Ann (2015)** both of them concluded that the convergence rate of Bisection technique is extremely unrushed and it is very hard to widen such type of equations. Thus by comparing, Newton Raphson method has a fast converging rate.[3]
- **B. Saheya, Guoqing Chen, Yunkang Sui and Caiying Wu (2016)** updated that the Newton Rahson method for solving of nonlinear conditions by reprocessing past iterative details. In the new technique, the capacity worth of the previous emphasis point was utilized for altering the Newton direction.[4]
- **Shuliang Huang, Arif Rafiq, Muhammad Rizwan Shahzad and Faisal Ali (2018)** presented two advanced iterative methods with a framework for solving nonlinear equations. The orders of convergence of our planned methods are four and five and thus the efficiency increases.[5]
- **Ekta Sharma, Sunil Panday and Mona Dwivedi (2020)** considered new optimal fourth order iterative method. The primary idea of this method is to achieve successfully higher

order convergence. Each iteration requires one function evaluation and two first derivative evaluations [6].

History

Mathematical strategies are very old as Egyptian Rhind Papyrus, these addresses a value discovering strategy for finding out root. Old Greek mathematicians made a lot of extra advancement in mathematical strategies. While there is slight existing information on the extension the Bisection method, we can find that it was built up a short time following the Intermediate Value Theorem was first demonstrated by Bernard Bolzano in 1817.The Regula False position technique was present in papyri from early Egyptian mathematics cuneiform and in tablets from early Babylonian mathematics. The Newton Raphson technique is imitative since Isaac Newton's portrayal of a specific instance of the strategy in his edition. Newton applied the technique for the polynomials through an underlying value approximate and takes out a progression of blunder remedies. The Newton Raphson technique was earliest given in 1685, he put the technique for polynomials and however, the technique was abstained. [4]

Bisection Method

Bisection method is also known as Bolzano technique for finding out root of the given non linear equation depend on the repeated applications of intermediate value theorem. [5]

Strategy:

Let the approximate value of root be $x_1 = \frac{h+g}{2}$, Now if we evaluate $f(x_1)$ there are three possibilities:

1) $F(x_1) = 0$, x_1 is the root.
2) $F(x_1) < 0$, the root lies in the interval (x_1, g).
3) $F(x_1) > 0$, the root lies in the interval (h, x_1)

In case (1), if we presume that there is one root only, then the process will be ended. If case (2), (3) take place then due to Bisection process the interval can be repeated till the precise root is obtained. [3].

Order of convergence:

In Bisection method, the new interval is obtained at each iteration whose length is identical to the not whole of the interval span obtained in previous iteration. Since we take the midpoint of intervals as the successive approximation so the upper bound of the error in any approximation is the half of the error in previous approximation

i.e. $|e_{k+1}| \leq \frac{1}{2}|e_k|, \qquad k = 0,1,2,3 \dots \dots \dots$

So, Bisection technique order is one i.e. it converges linearly with rate ½ or at each step, the error decreases by a factor of ½. [3].

False Position Method

False position method is also called as method of linear interpolation or Regula Falsi position method. It stays a powerful option in contrast to the Bisection technique for resolving non linear equation (f(x) = 0) whose root lies in the interval (h,g) such that f is continuous on the interval [h,g] and f(h) and f(g) have inverse signs[4].

Strategy:

Now we connect the two points (h, (f (h)) and (g, f (g)) on the curve y = f(x) by a straight line segment.

The equation of the chord connecting the points (h, (f (h)) and (g, f (g)) is given by

$$y - f(h) = \frac{f(g) - f(h)}{g - h}(x - h)$$

$$0 - f(h) = \frac{f(g) - f(h)}{g - h}(x - h)$$

$$x = h - \frac{g - h}{f(g) - f(h)} f(h)$$

$$x = \frac{hf(g) - gf(h)}{f(g) - f(h)}$$

Hence the first approximate value is given by

$$x_1 = \frac{hf(g) - gf(h)}{f(g) - f(h)}$$

False position method possibilities are similar to Bisection method.

Order of Convergence

The general iterative formula for False Position method is

$$x_{s+1} = x_s - \frac{x_s - x_{s-1}}{f(x_s) - f(x_{s-1})} f(x_s) \ldots \ldots . (1)$$

If e_{s-1} ,e_s and e_{s+1} are the errors in the approximations x_{s-1} ,x_s and x_{s+1} respectively then

$e_{s-1} = x_{s-1} - £$, $e_s = x_s - £$ and $e_{s+1} = x_{s+1} - £$

Substituting these values in (1) ,

$$£ + e_{s+1} = £ + e_s - \frac{(£ + e_s) - (£ + e_{s-1})}{f(£ + e_s) - f(£ + e_{s-1})} f(£ + e_s)$$

$$e_{s+1} = e_s - \frac{(e_s - e_{s-1})f(£ + e_s)}{f(£ + e_s) - f(£ + e_{s-1})} \ldots \ldots \ldots \ldots (2)$$

Expanding $f(£ + e_s)$ and $f'(£ + e_{s-1})$ using taylor series about the point £ and substituting f(£) = 0 in (2)

$$e_{s+1} = e_s - \frac{e_s f'(£) + \frac{1}{2} e_s{}^2 f''(£) + \cdots .}{f'(£) + \frac{1}{2}(e_s + e_{s-1}) f''(£) + \cdots}$$

$$= e_s - \left[e^s + \frac{1}{2} e_s{}^2 \frac{f''(£)}{f'(£)} + \cdots\right]\left[1 - (e_s - e_{s-1})\frac{f''(£)}{f'(£)} + \cdots\right] \text{ by binomial theorem}$$

$$= \frac{1 f''(£)}{2 f'(£)} e_s e_{s-1} + \cdots terms\ having\ higher\ order\ of\ errors$$

$e_{s+1} = \frac{1 f''(£)}{2 f'(£)} e_s e_{s-1}$ ..... (3) (Neglecting the higher order term)

Equation (3) in terms of error is called error equation

$$e_{s+1} = \frac{1 f''(£)}{2 f'(£)} e_0 e_s$$

e_{s+1} = Ae_s .... (7)

A = $\frac{1 f''(£)}{2 f'(£)} e_0$ *it is an asymptotic constant*

From equation (7) we can find out that the Falsi technique is linear order convergent i.e. order of convergence of this method is 1 [4], [5].

Newton Raphson Methods

Newton Raphson formula converges given that the initial estimation x_0 is selected adequately nearer to the root. In the starting, we suppose two numbers r and s such that f (r) and f (s) are of inverse signs. Hence the first approximate root lies between r and s.

Strategy:

Suppose x_0 be an estimated and let h = £ - x_0

Since £ is the accurate root of the equation

F (£) = 0 or f(x_0 + h) .... (1)

By using second degree terminated Taylor expression of f(x_0 + h) about x_0, we have

$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0 + \theta h), 0 < \theta < 1$ ...... (2)

Using (1) and (2)

$$h = -\frac{f(x_0)}{f'(x_0)}$$

A better approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

In general, starting with the approximation x_k, the next approximation x_{k+1} is given by

$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$; k = 0,1,2 ....(3)

The formula (3) is called Newton Raphson formula.

Order of convergence

Let us assume that £ is a simple root of the equation f(x) = 0

The general Newton Raphson formula is

$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$.........(1)

If e_k and e_{k+1} are the errors in the approximations x_k and x_{k+1} respectively then

$e_k = x_k$ - £ and $e_{k+1} = x_{k+1}$ - £

i.e. x_k = £+ e_k and x_{k+1} = £+ e_{k+1}

Substituting these values in (1),

$$£ + e_{k+1} = £ + e_k - \frac{f(£ + e_k)}{f'(£ + e_k)}$$

$$e_{k+1} = e_k - \frac{f(£ + e_k)}{f'(£ + e_k)}$$

Expanding $f(£ + e_k)$ and $f'(£ + e_{k-1})$ using Taylor series about the point £ and substituting F (£) = 0 we get

$$e_{k+1} = e_k - \frac{e_k + \frac{1}{2}{e_k}^2 f''(£) + \cdots.}{f'(£) + e_k f''(£) + \cdots}$$

$$e_{k+1} = e_k - \frac{e_k + \frac{1}{2}{e_k}^2 f''(£) + \cdots.}{f'(£) + e_k f''(£) + \cdots}$$

$= e_k - \left[e^k + \frac{1}{2}{e_k}^2 \frac{f''(£)}{f'(£)} + \cdots\right][1 - (e_k)\frac{f''(£)}{f'(£)} + \cdots]$ by binomial theorem

$e_{k+1} = \frac{1}{2}{e_k}^2 \frac{f''(£)}{f'(£)}$ [By neglecting the terms containg higher powers of error]

$e_{k+1} = c{e_k}^2$ .... (2)

Where C $= \frac{1}{2}\frac{f''(£)}{f'(£)}$

From equation (2) it is clear that the Newton Raphson method is second order convergent. [5]

Results and Findings

- Convergence is assured in the bisection method for any f(x) which is continuous in the interval containing the root.
- By comparing the results, False Position technique has guaranteed convergence and it converges quicker as compared to Bisection technique.[5]
- In a nutshell, Newton Raphson is fastest because it is second order convergent i.e order of convergence is 2 whereas, Bisection and False Position method are linearly convergent i.e order of convergence is 1.

- In Bisection method and False position method each iteration involves only one function evaluation so computational effort is less whereas Newton Raphson Method requires two function evaluation per iteration. [2]
- In False position technique, convergence can have moderate speed because all the estimations of the root can be lie on the same side.
- Bisection technique needs a huge number of iterations to obtain a precision for the root. By comparing we can say that the Bisection method is rather slow.[4]
- The Bisection technique is extremely thoughtful to the initial values. The Falsi technique is slight thoughtful to the intermediate terms. However Newton Raphson technique is extremely thoughtful to the initial terms.
- Bisection technique requires very less measure of endeavours in computational labour as it is the easiest of all the iterative strategy. Falsi technique needs more measure of computational labour per iteration which is equivalent to one function assessment only. However Newton Raphson technique needs significant amount of attempts and time in calculation of the values of f(x) and f '(x).[5]
- It is understandable from Newton Raphson formula that the larger the numerical value of derivative f '(x), the lesser is the correction that must be done to get the correct value of the root, If the sketch of the curve y= f(x) is almost vertical and when it crosses the x-axis, the accurate value of the root can be created with huge speed and a small labour [5], [6].

Drawbacks of Newton Raphson Method

- Newton Raphson is not applicable to solve the equations whose sketch is nearly horizontal when it crosses the x-axis as it would make f '(x) zero. So in that case method fails.
- Newton Raphson method is very sensitive to the choice of initial approximation. If the initial approximation is not properly selected then this method can fall in an endless loop and might even fail altogether in some cases.
- This method requires two function evaluations per iteration. So the computation per iteration is more compared to other methods.
- Newton Raphson Method is difficult to apply when the derivative of f(x) is not a simple expression.[4],[5]

Despite of these drawbacks, it is the speed of convergence for which Newton Raphson Method is called the best method among Bisection and False Position method.

These drawbacks have allowed presenting the most recent research results in the field of iterative techniques for solving nonlinear equations.

Iterative methods developed in recent years to overcome drawbacks:

New optimal fourth order iterative technique [7] is the recently developed method in which each iteration needs one function evaluation and two first derivative evaluations and hence the efficiency of this method is 1.5874. The above mentioned technique is fourth order convergent. It includes finding the root of nonlinear equations at the small computational cost of the first derivative of the function. This method is developed as multipoint iteration method which is without memory based on n-evaluations, it can attain best possible convergence order 2^{n-1}. This method had successfully achieved higher order convergence due to which it is the fastest method (iterative steps are less) [7]. In another method, new advanced order iterative methods are developed [8] , the researchers have constructed a few new iterative techniques whose convergence order is four and five with the help of auxiliary function based on modified HPM (homotopy perturbation method)[8]. Furthermore, the multiplicative methods converge quicker contrast to the ordinary root finding techniques for solving equations. [9]

Future Scope

The recent researches can be used for developing iterative methods for multiple roots and build some new methods. In coming years, we aim to increase the higher order convergence method which includes memory version via self accelerating parameters.

Conclusion

Iterative methods are used to provide constructive approximation to find out the roots of nonlinear equations. This study aimed to compare iterative methods and which one is most preferred among them. The study concludes that the greater the rate of convergence determines how fast solution of the equation can be obtained. Hence, Newton Raphson method is the best technique for computing the roots of nonlinear equation. Despite its merits it has its own demerits due to which numerous iterative techniques have been developed in recent years. In this paper we have discussed three methods which are developed recently.

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