Comparative Studyof Different Central Difference Interpolation Formulas

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Abstract: When taking care of issues utilizing numerical strategies it is generally important to set up a model and to record conditions communicating the limitations and actual laws that apply. These conditions should now be tackled and a decision introduces itself. One route is to continue utilizing regular strategies for math, getting an answer as a recipe, or set of formulae. Another strategy is to communicate the conditions so that they might be addressed computationally, i.e. by utilizing strategies for mathematical examination or Numerical analysis. In this paper we are aimed to discuss interpolation, various methods to solve central difference interpolation, their generalizations. Applications of interpolation are also discussed and one can easily understand the concepts of the paper.

Keywords: Interpolation, central difference, Gauss forward, Gauss backward, Stirling Formula, Bessel's formula.

1.Introduction:

Sometimes, we have to compute the value of a dependent variable for a given independent variable, but the explicit relation between them is not known. For example, we are available with the population of India for the years 1951, 1961, 1971, 1981,1991 and 2001. There is no exact mathematical expression available that will give the population for any given year. So, if it is asked to find the population of India in the year 2000, one cannot determine the population for the given year analytically. But by using Interpolation, one can determine the population approximately for any year.

Many researchers gave their views on Interpolation and the different methods of central difference interpolation formulas. Some of them are discussed as under:

Akima [1970], evolved a new mathematical method in which she takes a set of numbers in a plane and tries to fit a smooth curve to them. The research aims to take a set of values and develop a method in which the resultant curve is flat and original. In this approach of interpolation, the curve's steepness for each given set of values was determined locally and the polynomial representing the curve between the given set of values was evaluated by the coordinates of the slope at the given set of values. After comparison, it is concluded that the resultant curve acquired by the new method was almost similar to the curves were drawn by hand, which are either drawn by other mathematical methods.

Atkinson (1989), Carl and Boor (1980)- gave their views regarding Newton's divided difference formula for unevenly spaced points.Burden and Faires (2001) and Suli and Mayers (2003) gives an idea in relation to the Lagrange formula for interpolation. The given two methods are very useful to determine the polynomial interpolation for any random degree with an infinite number of sets of values.

Abdulla et al. (2004) presented a very efficient formula for central difference interpolation. They put forward this formula from gauss backward interpolation formula from Gauss's backward interpolation formula by subtracting one unit from the subscripts in Gauss's forward formula and replacing the component u with u+1. They also showed the similarities between the prevailing and the newly developed formulas based on differences. Results of the paper reveal that the new approach is well organized and gives better results for calculating the functional values between a given set of values.

Liu et al. (2006), made a study on the finite difference method (FRDM) and presented a new method of radial point interpolation. In this method, they assumed the least-square technique, which results in a system matrix that shows better properties such as symmetry and positive correctness. In their new strategy, to achieve both irregular domain and radial points interpolation, a convolutional finite difference approach is used with a local irregular node technique with versatility and solution constancy, which are both lacking in most collection approaches.

Reuter et al. (2007), made a study for the Shuttle Radar Topography Mission based on the void-filling interpolation approach. In their research, they discovered that the void-filling algorithm depends upon the size and terrain type of data. For approving theirresearch they took 1304 fake but realistic speech samples from six terrain types and eight void size classes.

Muthumalai(2008) -based on Neville's and Aitken's concepts on an algorithm, Muthumalai studied new iterativeinterpolation,numerical differentiation, numerical integration formulas for uniformly and irregularly spaced data.

Singh andBhadauria (2009), proposed a Finite Difference Formulae for unequal Sub-interval by using Lagrange’s interpolation formula. They used the concept of Lagrange’s interpolation and unequally spaced grid points for finding the general finite difference formulae and the error terms associated with that. As a special case of study, the finite difference methods and error terms forsub-intervals that are equally spaced were achieved in their research.Interpolation was used by Bater et al. (2009) for calculating the errors related to the lidar-derived Digital evaluation model. They discovered from their study that lidar technology is capable of measuring a wide range of vegetation matrices, whose estimates are typically based on the relative heights above a Digital Evaluation Model (DEM. For this, they tested seven interpolation routines on the small footprint lidar data collected over the vegetation classes on Vancouver Island.

Garnero and Godone (2013) studied digital terrain models which are used for ecological and land-related applications by considering multiple interpolation techniques.Muthumala and Uthra(2014) investigated both Newton's and Lagrange's interpolation formula of divided differences and proposed a new interpolation formula. And after doing the comparative study of all the existing and the newly proposed methods, it is concluded that the improved formula is more efficient and accurate.

Srivastav et al. (2015) discussed four different interpolation methods, namely Newton-Forward, Newton-Backward, Lagrange, Newton- divided difference, for solving the real-life problem. Das and Chakrabarty (2016) applied Lagrange's interpolation method to construct a formula that was needed to generate numerical data for India's total population. Other methods for the same purpose were discovered as part of the effort.

Moheuddin et al. (2019) developed a method of central difference interpolation based on the mixture of Gauss’s third formula (formed by advancing the subscript of Gauss’s backward formula with one unit and changing the factor u with u-1), Gauss's backward formula, Gauss's forward formula. They also give a graphical presentation as well as a comparison through all the existing interpolation formulas with their profound method of central difference interpolation. By the comparison and graphical presentation, the new method gives the best result with the lowest error from another existing interpolation formula.

Roseline et al. (2019): In this paper, theydeveloped a central difference interpolation formula obtained from Gauss's backward formula and another formula by advancing the subscript of Gauss's forward formulawith two units and u was substituted with u+2. They also compare the proposed interpolation formula to known interpolation formulas (Stirling's formula, Bessel's formula) based on differences, and utilize mathematicalnorm concepts to verify their research.

The main purpose of this paper is to examine the different central difference formulas with their workings, origination and also do a comparative study of all the formulas through some examples.This paper is very basic and can be very useful for UG as well as PG students to continue their research. Everything is explained in a very clear manner.

2. Some basic definitions:

In this section, some basic terms regarding interpolation have been discussed:

2.1 Interpolation:

"Interpolation is the art of reading between the lines of the table", says Thiele.

It also refers to the addition or filling up of middle terms of the series. Hence, Interpolation is thetechnique of approximating the function value for any undetermined independent variable, whereas Extrapolation is the technique of computing the function value outside the stated range.

2.2 Premise of interpolation:

For interpolation to work, certain premises must be fulfilled:

1. During the period under examination, there are no significant changes in the values.
2. Values should increase and fall consistently. For example, if the given data is on the number of fatalities in various years in a particular town, and some of the observations are for years when the town was affected by war or epidemic, interpolation approaches are ineffective.
3. In the methodology of finite differences, when a given set of values is expressed in polynomial form, it is observed that if values of the function are given in a clear and verified manner then one can easily compute 'y' in accordance with ' x '.

And if the value of the function is not known, then one will introduce some simpler function. For eg. $\emptyset(x)$ such that both the functions have the same set of values. This process is what someone called Interpolation.

Then after that, there comes a major term that has its importance in interpolation i.e. Finite differences.

2.3 Finite difference function: The math of finite differences gives out the changes that occur in the value of the dependent variable by the finite changes in the independent variable.

So, there are have three types of differences in the math of finite differences:

Let us consider a function x =g(y) and tabulate it for the uniformly spaced values y = y_0, y_0+h, y_0+ 2h……………. y_o+nh, given, x=$x_0, x_1, x_2 \dots \dots \dots .. x_n$. To find the values of f(y) and f'(y), for some middle values of x, the given three differences are helpful:

1. **Forward difference:** The given set of values such as $x_0 - x_1$, $x_2 - x_1$, $x_3 - x_2 \dots \dots \dots .. x_n - x_{n-1}$ when expressed by Δx_0, Δx_1, $\Delta x_2 \dots \dots \dots \dots \dots . \Delta x_{n-1}$ respectively are known as first forward differences and the symbolΔ is known as the forward difference operator.
2. **Backward difference:** The given set of values such as $x_1 - x_0$, $x_2 - x_1$, $x_3 - x_2$, $\dots \dots \dots \dots \dots . x_n - x_{n-1}$ when expressed by ∇x_1, ∇x_2, $\nabla x_3 \dots \dots \dots \dots \dots .. \nabla x_n$ respectively, are known as first backward differences and the symbol $\nabla$is known as the backward difference operator.
3. **Central difference**: The relation that defines the central difference operator 'd' is given as :

$$x_1 - x_0 = \delta x_{\frac{1}{2}},\ \ x_2 - x_1 = \delta x_{\frac{3}{2}}, \dots \dots \dots . x_n - x_{n-1} = \delta x_{n-\frac{1}{2}}$$

3. Study of different central difference Interpolation formulas:

It is easy to calculate the value of y corresponding to any value of x if the value of the function g(x) is known. On the other hand, if one is unfamiliar with the value of g(x), then it is quite difficult to find the solution of g(x) by using a tabulated collection of numbers (x_i, y_i).

In such circumstances, simply substitute a normal function $\emptyset(x)$ for g(x) which shows the same values as that of g(x) at the given tabulated collection of numbers. As a result, one can find any other value from $\emptyset(x)$ which is called as an interpolating or smoothing function

The methodology of finite differences helps in understanding the concept of Interpolation.By using forward and backward interpolation differences of a function the two fundamental interpolation formulae can be obtained. These formulas are frequently used in engineering and scientific research. Before discussing central difference interpolation formulas let's take a small look at the forward and backward interpolation formulas:

3.1 Newton's forward interpolation formula: let us consider a function x = g(y), take the values $x_0, x_1, x_2, x_3, x_4 \dots \dots \dots \dots \dots x_n$in accordance with the values $y_0, y_1, y_2, y_3 \dots \dots .. y_n$ of y. Let these values of y to be stated in a spaced manner such that $y_j = y_0 + jh$ (j =0, 1, 2……..). assuming x(y) to be an n^{th} degree polynomial in y such that x $(y_0) = x_0$, $x(y_n) = x_1$, ……..... $x(y_n) = x_n$, we can write,

$$x_p = x_0 + p\Delta x_0 + \frac{p\,(p-1)}{2!}\Delta^2 x_0 + \frac{p(p-1)\,(p-2)}{3!}\Delta^3 x_0 + \cdots \dots \dots \dots \dots \dots \dots (1)$$

This method is beneficial for determining values of x at the beginning of a tabulated set of values and extrapolating values of x a bit backward (i.e. to the left) of x_0.

One more result is there: The linear interpolation is given by the first two terms of this formula, while the parabolic interpolation is given by the first three terms, and so on.

3.2 Newtons backward interpolation: let us consider a function x= g(y), take the values x_0, x_1, x_2 ... ...in accordance with the valuesy_0, y_0+h, y_0+2h......... of y. Consider the problem of determining the values of g(y) for y = $y_n + ph$, where p may be any real number. Then we have:

$$x_p = x_n + p\nabla x_n + \frac{p(p+1)}{2!}\nabla^2 x_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 x_n + \cdots \ldots\ldots\ldots\ldots\ldots (2)$$

It is termed Newton's backward interpolation formula as the given equations consist of x_n and the backward differences of x_n.

This method is beneficial for determining values of x at the end of the tabulated set of values and for extrapolating values of x a little ahead (to the right) of x_n.

3.3 Central difference interpolation: So, in Newton's backward and forward difference interpolation formula,it is learned that they can be used for interpolation around the beginning and end of a table of data. Now let's introduce the central difference interpolation formula that is best applicable for determining the values near the middle of the table.

In the central differenceinterpolation, there are four methods, that are required to formulate the central difference:

3.3.1 Gauss Forward interpolation formula: If y takes the values $y_0-2h,\ y_0-h, y_0,\ y_0+h,\ y_0+2h$ andcorresponding values of x = g(y) are $x_{-2},\ x_{-1},\ x_0,\ x_1,\ x_2,$ then the formula for Gauss forward interpolation is:

$$x_p = x_0 + p\Delta x_0 + \frac{p\ (p-1)}{2!}\Delta^2 x_{-1} + \frac{(p+1)p\ (p-1)}{3!}\Delta^3 x_{-1} + \frac{(p+1)p(p-1)\ (p-2)}{4!}\Delta^4 x_{-2} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (3)$$

As seen below, this formula uses odd differences slightly below the central line,and even differences are taken on the central line:

$\longrightarrow$ $Central\ line x_0 \Delta^2 x_{-1} \Delta^4 x_{-2}\ \ \Delta^6 x_{-3}$

$\Delta x_0 \Delta^3 x_{-1} \Delta^5 x_{-2}$

This method is beneficial to estimate values of x for p (0<p<1), calculated in a forward direction from the origin.

When 'p' is between 0 and $\frac{1}{2}$, this formula is applied.

3.3.2 Gauss backward interpolation: Gauss backward interpolation formula is derived by modifying Newton's forward interpolation formula. It's beneficial for interpolating x values for a negative 'p' value that's between -1 and 0 or in terms of central difference notation the values lie between -1/2 and 0.

The gauss backwardformula is written as:

$$x_p = x_0 + p\Delta x_{-1} + \frac{p\,(p+1)}{2!}\Delta^2 x_{-1} + \frac{(p+1)p\,(p-1)}{3!}\Delta^3 x_{-2} + \frac{(p+2)(p+1)p\,(p-1)}{4!}\Delta^4 x_{-2} \dots\dots\dots\dots\dots\dots\dots\dots\dots.(4)$$

As illustrated below, this approach has odd differences above the central line , and even differences are taken on the central line:

$\Delta x_{-1} \Delta^3 x_{-2} \Delta^5 x_{-3}$

Central line: $x_0 \Delta^2 x_{-1} \Delta^4 x_{-2}$

This is the rule formulation of gauss backward interpolation.

The Gauss forward and backward interpolation formulas aren't very useful. These, on the other hand, might be considered as the first phase in attaining the crucial formula for the next sections:

3.3.3 Stirling's central difference interpolation formula: After taking the arithmetic mean of Gauss forward and backward interpolation we will obtain Stirling's Interpolation formula. This formula is used for an odd number of equally spaced values.

So, aftertaking the mean of Gauss forward and backward interpolation we get Stirling's formula as:

$$x_p = x_0 + p\left[\frac{\Delta x_0 + \Delta x_{-1}}{2}\right] + \frac{p^2}{2!}\Delta^2 x_{-1} + \frac{p(p^2-1)}{3!}\left[\frac{\Delta x_{-1} + \Delta x_{-2}}{2}\right] + \frac{p^2(p^2-1)}{4!}\Delta^4 x_{-2} + \cdots \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.(5)$$

The means of odd differences just above and below the central line and even differences onthe central line are used in this formula, as illustrated below:

$\Delta x_{-1} \Delta^3 x_{-2}$

Central line: $x_0 \Delta^2 x_{-1} \Delta^4 x_{-2}$

$\Delta x_0 \Delta^3 x_{-1}$

By applying this format one can make up Stirling's formula an easy way to remind the terms in the formula.

Stirling's interpolation formula gives the best approximate result when 'p' lies between $\frac{-1}{4}$ and $\frac{1}{4}$.

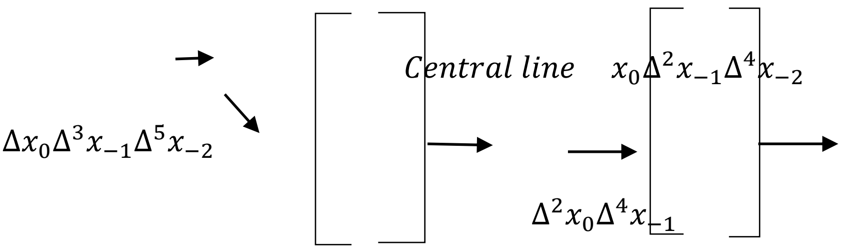
3.3.4 Bessel's central difference interpolation formula: This central difference formula is acquiredafter taking the arithmetic mean of Gauss's forward and backward interpolation formula. But one difference is that the backward formula is taken after one modification.

The modification is that, take the gauss backward formula after replacing' p ' with (p-1) and add 1 to each argument in the formula. Then after replacing the factor take mean of both Gauss's forward and backward formula and the resultant will be Bessel's formula of interpolation.

The formula is:

$$x_p = x_0 + p\Delta x_0 + \frac{p\,(p-1)}{2!}\left[\frac{\Delta^2 x_0 + \Delta^2 x_{-1}}{2}\right] + \frac{\left(p-\frac{1}{2}\right)\,p\,(p-1)}{3!}\Delta^3 x_{-1} \dots\dots\dots..(6)$$

For practical applications, this is a very useful formula. As illustrated below, it comprises odd differences below the central line, as well as means of even differences of and below this line.



Bessel's method is best applicable when 'p' is between $\frac{1}{4}$ and $\frac{3}{4}$ and is used often when the interpolating point lies near the middle of the table and the number of arguments in the problem is even.

4. Numerical discussion:

4.1. Consider a function x = $4y^2 + y + 1$, and value of x for equidistant spaced values of y are:

y	**1**	**3**	**5**	**7**	**9**	**11**	**13**
x	**6**	**40**	**106**	**204**	**334**	**496**	**690**

Solution: let us solve this problem with all the four central difference interpolation formulae:

Here let us find this for y=6, for that, we have $y_0 = 7$ and h=2 and

p=$\frac{x-x_0}{h} = \frac{6-7}{2} = -0.5$

The central difference table for the given problem is as shown below:

y	x	Δx	$\Delta^2 x$	$\Delta^3 x$
1	6			
		34		
3	40		32	
		66		0
5	106		32	
		98		0
7	204		32	
		130		0
9	334		32	
		162		0
11	496		32	
		194		
13	690			

Let us solve this problem by all the central difference formulae:

1. **Gauss forward interpolation formula:**

$$x(6) = 204 + (-0.5)(130) + \frac{(-0.5)(-0.5-1)}{2}(32)\ldots\ldots\ldots \quad [\text{ using } (3)]$$

$$= 204 - 65 + (-0.5)(-1.5)(16)$$

$$= 204 - 65 + 12 = 151$$

Hence we get x(6) = 151 by gauss forward interpolation formula.

2. **Gauss backward interpolation formula:**

$$x(6) = 204 + (-0.5)(98) + \frac{(-0.5)(-0.5+1)}{2}(32) \ldots\ldots\ldots\ldots \text{[using (4)]}$$

$$= 204 - 49 + (-0.5)(0.5)(16)$$

$$= 204 - 49 - 4$$

=151

3. **Stirling's formula:**

$$x(6) = 204 + (-0.5)\left[\frac{98+130}{2}\right] + \frac{(-0.5)^2}{2!}(32)\ldots\ldots\ldots\ldots \text{[using (5)]}$$

$$= 204 + (-0.5)(114) + 0.25(16)$$

$$= 204 - 57 + 4$$

=151

4. **Bessel's formula:**

$$x(6) = 204 + (-0.5)(130) + \frac{(-0.5)(-0.5-1)}{2}\left[\frac{32+32}{2}\right] \ldots\ldots\ldots \text{[using(6)]}$$

=204- 65+(0.375)(32)

= 151

Hence, by all the methods we get the same result as x (6) =151

4.2. The population of a town in a year is:

Year	1931	1941	1951	1961	1971
Population (in thousand)	15	20	27	39	52

Sol: let us solve this problem with all the four central difference interpolation formulae:

Here let us find this for the year=1946, for that, we have $x_0 = 1951$ and h=2 and

p=$\frac{x-x_0}{h} = \frac{1946-1951}{2} = -0.5$

Year	Population (in thousand)	Δx	$\Delta^2 x$	$\Delta^3 x$	$\Delta^4 x$
1931	15				
		5			
1941	20		2		
		7		3	
1951	27		5		-7
		12		-4	
1961	39		1		
		13			
1971	52				

1. **Gauss forward interpolation:**

$$x(1946) = 27 + (-0.5)12 + \frac{(-0.5)(-1.5)}{2}(5) \dots \dots \dots \dots . . [\text{using}(3)]$$

=27-6+1.875

=22.875

=22875 thousand

2. **Gauss backward interpolation**:

$$x(1946) = 27 + (-0.5)7 + \frac{(-0.5)(-0.5+1)}{2}(5) \dots \dots \dots . . [\text{using}(4)]$$

= 27-3.5-0.625

= 22.875

=22875 thousand

3. **Stirling's formula**:

$$x(1946) = 27 + (-0.5)\left[\frac{7+12}{2}\right] + \frac{(0.5)^2}{2}(5) \ldots\ldots\ldots\ldots [\text{using}(5)]$$

= 27-4.75+0.625

= 22.875

= 22875 thousand

4. **Bessel's formula**:

$$x(1946) = 27 + (-0.5)12 + \frac{(-0.5)(-0.5-1)}{2}\left[\frac{5+1}{2}\right] \ldots\ldots\ldots\ldots [\text{using}(6)]$$

= 27-6+1.875

=22.875

=22875 thousand

Hence, by all the methods we get the same result as x (1946) =22875 thousand

5.Alternatives for interpolation formulas:

So far, we have come to know about the several interpolation formulae likeNewton'sforward, Newton's backward, Gauss's backward and Gauss's forward, Stirling's formula, Bessel's formula for calculating $x_p$for equally spaced tabulated values.

Now we will do the comparative study of all of them that which formula or method is best applicable at which stage:

The central difference formula has smaller coefficients that converge faster than Newton's formula. The coefficients in Stirling's formula fall more rapidly than those in Bessel's formula after a few terms, while the coefficients in Bessel's formula decline more rapidly than those in Newton's formula after a few terms.As a result,wherever possible, we will prefer the central difference formula instead of Newton's formula.

The correct interpolation method, on the other hand, is determined by the position of the interpolated values in the given data. The following rules will assist you in comprehending all of the strategies.:

1. For calculating the tabulated set of values at the starting of the table, use Newton's forward interpolation formula.
2. For calculatingvalue nearby the end of the table,prefer Newton's backward interpolation formula.

3. For calculating value close to the center of the table, prefer either Stirling's formula or Bessel's formula.
   - If someone wants to interpolate a value of 'p' that is between $\frac{-1}{4}$ and $\frac{1}{4}$, prefer Stirling's formula.
   - Use Bessel's formula, if someone wants to interpolate for a value of 'p' between $\frac{1}{4}$ and $\frac{3}{4}$.
   - By reminding all these rules one can easily make the right choice that in which problem which method is best applicable.

6.Applications of Interpolation in Real life:

As we know, the method of finding values at unknown points using known values or sample points is known as interpolation. So, various practical uses of interpolation as follows:

1. The concept of interpolation can be used to estimate unknown parameters for any geographic point data, such as rainfall,elevation,noise levels, chemical concentrations, and so on.
2. Using interpolation techniques to zoom digital images: Image processing for low-resolution digital images is a very challenging problem nowadays. It is because of errors that occur in quantization and sampling. Zooming in on such images is extremely difficult. As a result, when zooming in low-resolution images, we use interpolation functions.
3. Interpolation is often used outside the domain of mathematics to scale images and transform the sample rate of digital signals.
4. Interpolation is helpful whenever you have to scale things up or down regularly.
   For example- We may know how much food costs for a 10person event, 50 person event, or 100 person event, but we need a precise estimate of how much catering will cost for 25 people or 75 people. That's what interpolation allows us to achieve.

These are some of the examples of interpolation in real life as well as in other disciplines.

7. Observation of the Study: Based on the observations and readings, it is concluded from this paper that Interpolation is mostly used to assist users, who may be scientists, photographers, engineers, or mathematicians, in determining what information may exist outside of their collected information. The comparative study of all the methods helps us to understand that how one can solve various problems by using the best suitable interpolation method. Or how to make the right choice that which formula gives the result of the given problem more accurately.

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