Estimating the Entropy of a Lomax Distribution under Generalized Type-I Hybrid Censoring

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Abstract:

The data are said to be censored when some important information about the subject's event time that are required to make a conclusion is not available to the practitioner. Censoring is said to be single Type-I censoring (or time censoring) when the experimental time is fixed and the number of observed failures is a random variable. In contrast, censoring is said to be single Type-II censoring (or failure censoring) when the number of observed failure is fixed and the experimental time is a random variable. A mixture of Type-I and Type-II censoring is called a single hybrid censoring scheme. The disadvantage of a hybrid censoring scheme is that there is a possibility that very few failures may occur before time. In that case, the efficiency of the estimator(s) might below. For this reason, So, Scientists proposed the generalized Type-I hybrid censoring as a modification of the hybrid censoring scheme. The reason behind the proposed modification is to fix the underlying disadvantages inherent in the hybrid censoring scheme.

In information theory, entropy plays a central role which measures the uncertainty associated with the cumulative distribution function. The concept of information entropy was introduced by Claude Shannon in his 1948 paper "A Mathematical Theory of Communication".

In this paper, we obtain the entropy estimate of a two-parameter Lomax distribution based on the first type of hybrid censoring scheme (HCS). The maximum probability estimates to the unknown parameters are extracted to the entropy estimate.

Keywords: Entropy, Lomax distribution, maximum likelihood estimation,Generalized type-I hybrid censoring.

1. Introduction

Every probability distribution has some kind of uncertainty associated withit and the entropy is used to measure this uncertainty. One of the importantterms in statistical mechanics is entropy. Furthermore, it is a perceived asmeasure of the randomness of a probabilistic system. The entropy was introduced by [13] as a measure of the information associated with arandom variable or a signal. Let X is a random variable with a probabilitydensity function (pdf) $f(x)$and cumulative distribution function (cdf) $F(x)$.[6] defined the differential entropy $H(X)$ of therandom variable X as:

$$H(X) = -\int_{-\infty}^{\infty} f(x)\, log\, f(x)\, dx.$$

A widely spread distribution would result in high entropy, although a narrowly spread distribution would have a very low entropy. Many authors worked on estimating entropy for different lifetime distributions. [3] developed the entropy of upper record values and provided several upper and lower bounds for this entropy by using the hazard rate function.

Lifetime data analysis is used to analyse data in which the time until the event is of interest, like time until tumour recurrence, time until cardiovascular death after some treatment and time until a machine part fails. Censoring is present when we have some information about a subject's event time, but we do not know the exact event time. If we are removing unfailed units from a test at a pre-specified time this is known as "Time censoring" or "type-I censoring". Type-I censoring can be described as follow: a randomly selected sample of n units is subjected of a life test under some environmental conditions. The life times of the sample units are assumed to be independent and identically distributed $(i.i.d.)$ random variables. The experiment is stop when a pre-specified time T. The data collected consist of $x_{(1)} < x_{(2)} < \cdots < x_{(r)}$, plus the information that $(n-r)$ items survive beyond the time of termination T, where r is the number of uncensored items. If, instead of terminating the experiment at a pre-specified time T, we terminate it after the r^{th} failure, where r is fixed in advance, this is type-II censoring scheme. The data collected looks like those above but r is now random.

Other types of censoring schemes have been suggested in the literature for example see [6]. Based on practical and cost consideration a number r and a time T are chosen to control the experiment. We terminate the experiment as soon as the number of failed units reached r or the time on experiment becomes T, see [2]. One of those schemes is the hybrid censoring schemes (HCS) which is a mixture of type-I and type-II censoring schemes. We select a random sample of n unit and this sample is subjected to a life test under identical environmental conditions.

The likelihood function for the hybrid censored data will be written as:

$$L(x) = \frac{n!}{(n-D^*)!}\prod_{i=1}^{D^*} f(x_{(i)})(1 - f(T_*))^{n-D^*},$$

where $T_* = min(T, x_r)$, and D^* denotes the number of units that would fail before the time T_*see [10]. The main disadvantage of this type the most of the inference results are obtained under the condition that the number of observed failures is at least one, and moreover thither may be very little failures occurring up to the pre-fixed time T. In that case the efficiency of the estimator(s) may be very low. Because of this alternative hybrid censoring scheme that would terminate the experiment at the random time $T_* = \min(T, x_r)$, has been proposed. It is called the type-I hybrid censoring scheme.

The Lomax distribution, which is actually the Pareto type-II, was defined by [12]. It is a heavy-tailed distribution. It has been used in many applications such as actuarial science, economics, and life testing problems in engineering, see [1], [7] and [9]. There are many distributions that have close relationships with Lomax like the generalized Pareto distribution, the F-distribution, the q-exponential distribution and the beta prime distribution. Also, there are many variants of Lomax distribution like gamma Lomax, McDonald Lomax and weighted Lomax see [9], [11]and [5]. Furthermore, one characteristic of Lomax distribution is that it plays an important role in information theory and it is of great importance in probability and statistics. [8] introduced and study a new distribution with three lifetime parameters called Lomax inverse force.

In this paper, we obtain the entropy estimate of a two-parameter Lomax distribution based on the generalized type-I (HCS). The maximum likelihood estimates of the unknown parameters are obtained and plugged in the entropy function to obtain the maximum likelihood estimate of the entropy. The approximate Fisher information matrix is obtained and simulation studies are performed to assess the performance of the estimates with different sample sizes.

2. Generalized type-I hybrid censoring scheme (G Type-I HCS):

Although HCS-I avoids the disadvantage of the usual type-II censoring of taking a very long time to end, it does not guarantee enough failures to make efficient inference. On the other hand, HCS-II guarantees a minimum of r failures, it may take a very long time to terminate. To capture the advantages of both schemes, [4] proposed the G Type-I and Type-II HCS. G Type-I HCS could be described briefly as follow: assume that n identical items are put on a life test at time point 0. Fix r_1, r_2 and T such that $r_1 < r_2 < n, r_1, r_2 \in (1,2,3, \dots, n)$ and $T \in (0, \infty)$, then we are faced with one of two situations:
1- If the r_1^{th} failure occurs before time T, end the experiment at $\min(X_{r:n}, T)$.
2- If the r_1^{th} failure occurs after time T, end the experiment at $(X_{r:n})$.

It is clear that this G Type-I HCS modifies the Type-I HCS by allowing the experiment to continue beyond time T if very few failures had been observed up to time point T. Under this scheme, we are guarantee minimum of r_1 failures and maximum r_2 failures, with the possibility of some number of failures in between. In this case, the likelihood functions for three different cases are as follows, see [4].

$$L(x) = \begin{cases} \frac{n!}{(n-r_1)!} \prod_{i=1}^{r_1} f(x_{(i)}) \left(1 - F(x_{(r_1)})\right)^{n-r_1}, D = 0,1, \dots, r_1 - 1, \\ \frac{n!}{(n-D)!} \prod_{i=1}^{D} f(x_{(i)}) \left[1 - F(T)\right]^{n-D}, D = r_1, r_1 - 1, \dots, r_2 - 1, \\ \frac{n!}{(n-r_2)!} \prod_{i=1}^{r_2} f(x_{(i)}) \left[1 - F(x_{(r_2)})\right]^{n-r_2}, \qquad D = r_2, \end{cases} \tag{1}$$

where D denotes a number of failures observed before time T. Then D is a discrete random variable with support $\{0,1, \dots, r\}$with pdf

$$P_\theta(D = j) = \begin{cases} \binom{n}{j} (f(T))^j \left(1 - F(T)\right)^{n-j}, \quad j = 0,1, \dots, r - 1 \\ 1 - \sum_{j=0}^{r-1} \binom{n}{j} (f(T))^j \left(1 - F(T)\right)^{n-j}, \qquad j = r \end{cases} \tag{2}$$

In this scheme, we have one of the following types of observations:

Case I :

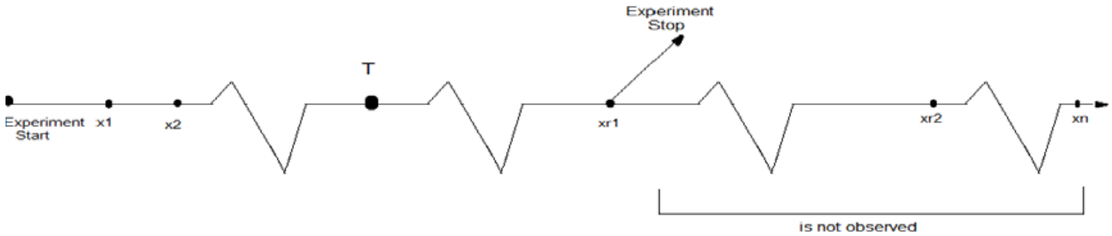

Figure 1. Schematic representation of the generalized type-I HCS.

Case II:

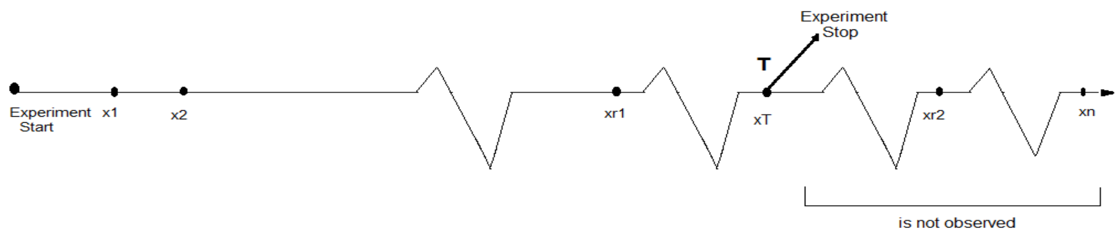

Figure 2. Schematic representation of the generalized type-I HCS.

Case III :

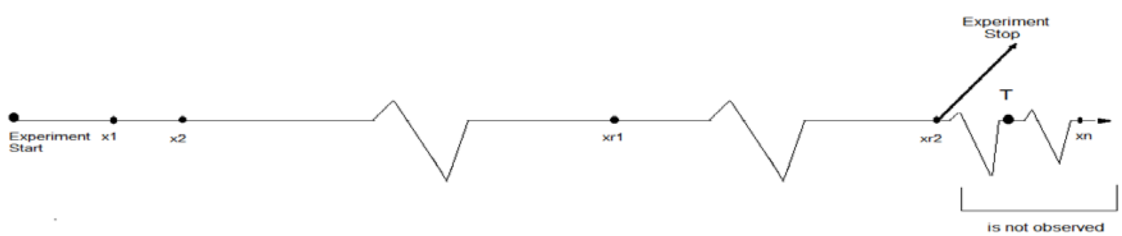

Figure 3. Schematic representation of the generalized type-I HCS.

3. The maximum likelihood estimation:

Let X be distributed as Lomax distribution with cdf $F(x)$ and pdf $f(x)$ defined respectively as:

$$F(x) = 1 - (1 + \frac{x}{\lambda})^{-\alpha}, \quad x>0 , \quad \alpha, \lambda > 0, \qquad (3)$$

and

$$f(x) = \frac{\alpha}{\lambda}(1 + \frac{x}{\lambda})^{-(\alpha+1)}, \qquad x > 0, \quad \alpha, \lambda > 0 , \qquad (4)$$

where λ is the scale parameter and α is the shape parameter. To obtain the maximum likelihood estimate we first construct the likelihood function which, of course, dependence in the sample scheme. In the complete sample, consider a life-testing experiment in which n identical units are put on test. Suppose n items were put in a life test and that we observe and record the failure times $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$ in the order in which they occur. Therefore $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$. If we can afford to wait until all items fail, we would have a complete sample and likelihood function would be:

$$L(\theta \,/\, x_{(i)}) = \prod_{i=1}^{n} f\left(x_{(i)}\right). \tag{5}$$

If we had a complete sample situation, we would have differentiated likelihood function in equation (5) with respect to the parameter of interest, equating these derivatives with zero and solving the "normal" equations to obtain the maximum likelihood estimates. Actually, we are interested in estimating the entropy of the distribution from which the complete sample data at hand came. The solution to this problem is readily available via the invariance principle of the maximum likelihood estimators which implies that the maximum likelihood estimate of the entropy is the entropy function of the MLE estimates of the parameters. That is

$$(\widehat{H(\theta)}) = H(\hat{\theta}). \tag{6}$$

In many situations in practice, the ideal situation in complete sample is violated and we are faced with what is known as the censoring problems. Censoring occurs when the observed values we hoped for are only partially known; at least for some observation. There are many reasons for censored data, some of which may be planed by the researcher, and therefore there are many types of censoring.

[2] give a good review of the history of censoring and of the many types of censoring. we are interested in the G Type-I HCS. So, let us assume that the lifetimes of the experimental unit are $i.i.d.$ Lomax random variables with pdf (4) and cdf (3). If d denotes the number of failures that occur by time point T, then based on the G Type-I HCS, the likelihood functions of α and λ are given by:

$$L = \frac{n!}{(n-r_1)!}\left(\frac{\alpha}{\lambda}\right)^{r_1} \prod_{i=1}^{r_1}\left(1+\frac{x_{(i)}}{\lambda}\right)^{-(\alpha+1)} \left(1+\frac{x_{(r_1)}}{\lambda}\right)^{-\alpha(n-r_1)} \tag{7}$$

The logarithm of (7) can be written as:

$$InL \propto r_1 In\alpha - r_1 In\lambda - (\alpha+1)\sum_{i=1}^{r_1}(1+\frac{x_{(i)}}{\lambda}) - \alpha(n-r_1)In\left(1+\frac{x_{(r_1)}}{\lambda}\right). \tag{8}$$

Taking derivatives with respect to α and λ of (8), and equality to zero, we obtain the following

$$\frac{\partial InL}{\partial \alpha}=\frac{r_1}{\alpha}-\sum_{i=1}^{r_1} In\left(1+\frac{x_{(i)}}{\lambda}\right)-(n-r_1)In\left(1+\frac{x_{(r_1)}}{\lambda}\right), \tag{9}$$

$$\frac{\partial InL}{\partial \lambda}=\frac{-r_1}{\lambda}+(\alpha+1)\sum_{i=1}^{r_1}\left(1+\frac{x_{(i)}}{\lambda\left(\lambda+x_{(i)}\right)}\right)+\alpha(n-r_1)\frac{x_{(r_1)}}{\lambda\left(\lambda+x_{(r_1)}\right)}. \tag{10}$$

$$\hat{\alpha}=\frac{r_1}{\sum_{i=1}^{r_1} ln(1+\frac{x_{(i)}}{\hat{\lambda}})-(n-r_1)\, ln(1+\frac{x_{(r_1)}}{\hat{\lambda}})} \tag{11}$$

Using (10) in (9) $\hat{\lambda}$ can be written as:

$$\hat{\lambda}=\frac{r_1}{(\hat{\alpha}+1)\sum_{i=1}^{r_1}(\frac{x_{(i)}}{\hat{\lambda}(\hat{\lambda}+x_{(i)})})+\hat{\alpha}(n-r_1)(\frac{x_{(r_1)}}{\hat{\lambda}(\hat{\lambda}+x_{(r_1)})})}. \tag{12}$$

Similarly, for case II and III in a G Type-I HCS, the estimate of α and λ can be written as:

$$\hat{\alpha}=\begin{cases}\dfrac{r_1}{\sum_{i=1}^{r_1} ln(1+\frac{x_{(i)}}{\hat{\lambda}})-(n-r_1)\, ln(1+\frac{x_{(r_1)}}{\hat{\lambda}})}, D=0,1,\dots,r_1-1,\\ \dfrac{D}{\sum_{i=1}^{D} ln(1+\frac{x_{(i)}}{\hat{\lambda}})-(n-D)\, ln(1+\frac{T}{\hat{\lambda}})}, D=r_1,r_1-1,\dots,r_2-1,\\ \dfrac{r_2}{\sum_{i=1}^{r_2} ln(1+\frac{x_{(i)}}{\hat{\lambda}})-(n-r_2)\, ln(1+\frac{x_{(r_2)}}{\hat{\lambda}})}, \qquad D=r_2.\end{cases} \tag{13}$$

And

$$\hat{\lambda}=\begin{cases}\dfrac{r_1}{(\hat{\alpha}+1)\sum_{i=1}^{r_1}\frac{x_{(i)}}{\hat{\lambda}(\hat{\lambda}+x_{(i)})}+\hat{\alpha}(n-r_1)\frac{x_{(r_1)}}{\hat{\lambda}(\hat{\lambda}+x_{(r_1)})}}, D=0,1,\dots,r_1-1,\\ \dfrac{D}{(\hat{\alpha}+1)\sum_{i=1}^{D}\frac{x_{(i)}}{\hat{\lambda}(\hat{\lambda}+x_{(i)})}+\hat{\alpha}(n-D)\frac{T}{\hat{\lambda}(\hat{\lambda}+T)}}, D=r_1,r_1-1,\dots,r_2-1,\\ \dfrac{r_2}{(\hat{\alpha}+1)\sum_{i=1}^{r_2}\frac{x_{(i)}}{\hat{\lambda}(\hat{\lambda}+x_{(i)})}+\hat{\alpha}(n-r_2)\frac{x_{(r_2)}}{\hat{\lambda}(\hat{\lambda}+x_{(r_2)})}}, \qquad D=r_2.\end{cases} \tag{14}$$

Equation (13) and (14) are very hard to evaluate theoretically and a numerical procedure is needed to solve these equations numerically. Mathematica 11 will be used to obtain the MLE of shape and scale parameter $\hat{\alpha}$and $\hat{\lambda}$of α and λ respectively.

For the pdf (4), entropy simplifies given by:

$$H(X) \equiv H(f) = -\int_{-\infty}^{\infty} f(x) log f(x) dx$$

$$= -\int_{-\infty}^{\infty} \frac{\alpha}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\alpha+1)} \log \frac{\alpha}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\alpha+1)} dx$$

$$= -\log\frac{\alpha}{\lambda} - (\alpha+1)\int_{-\infty}^{\infty} \frac{\alpha}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\alpha+1)} \log\left(1+\frac{x}{\lambda}\right) dx$$

$$= -\log\frac{\alpha}{\lambda} + (\alpha+1)\, E[\log\left(1+\frac{x}{\lambda}\right)]$$

On substituting $1+\frac{x}{\lambda} = t$and solving it of t, we get

$$E[\log\left(1+\frac{x}{\lambda}\right)] = \frac{1}{\alpha},$$

hence

$$H(x) = -\,log\frac{\alpha}{\lambda} + \left(\frac{\alpha+1}{\alpha}\right). \tag{15}$$

Once we obtain the MLE of α say $\hat{\alpha}$, and MLE of λ say $\hat{\lambda}$, the MLEs of entropy are obtained as:

$$\hat{H}(x) = -\,log\frac{\hat{\alpha}}{\hat{\lambda}} + \left(\frac{\hat{\alpha}+1}{\hat{\alpha}}\right). \tag{16}$$

The asymptotic variance-covariance matrix of $(\hat{\alpha}, \hat{\lambda})$ is obtained by inverting the information matrix with elements that are negatives of expected values of the second order derivatives of logarithms of the likelihood function,

$$\hat{I}(\hat{\alpha}, \hat{\lambda}) = -\begin{vmatrix} \frac{\partial^2 \ln L}{\partial \alpha^2} & \frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} & \frac{\partial^2 \ln L}{\partial \lambda^2} \end{vmatrix}_{\alpha=\hat{\alpha}}.$$

The element of Fisher information matrix are given as follows:

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = \frac{-r_1}{\hat{\alpha}},$$

$$\frac{\partial^2 \ln L}{\partial\alpha\partial\lambda} = \sum_{i=1}^{r_1} \frac{x_{(i)}}{\hat{\lambda}^2+x\hat{\lambda}} + (n-r_1)\frac{x_{(r_1)}}{\hat{\lambda}^2+x\hat{\lambda}},$$

and

$$\frac{\partial^2 \ln L}{\partial\lambda^2} = \frac{-r_1}{\hat{\lambda}^2} - \sum_{i=1}^{r_1}\left(1+\frac{x_{(i)}\left(2\hat{\lambda}-x_{(i)}\right)}{\hat{\lambda}^2\left(\hat{\lambda}+x_{(i)}\right)}\right).$$

4. Simulation study:

In this section, we present the results of a simulation study that was carried out to assess the performance of MLE estimation of the Lomax entropy. The assessment is carried out through measures of the bias, relative mean square error (RMSE), entropy and RMSE of Entropy under different variation of the G Type-I HCS. The simulation is carried out for different combination of n, r_1, r_2 and T values. In each case process is replicated N = 10000 times for a particular set of G Type-I HCS. The MLEs for the entropy were obtained as described before. We were able to express α in terms of λ as in formula (13) therefore obtaining the MLE estimates is attained be solving the equation (14). The computational system Mathematica 11 was used to solve equation (14) in λ. We substitute these values in (13) to obtain the values of α. These values of α and λ constitute their maximum likelihood estimates. We substitute these values in (16) to obtain the MLE estimates of the entropy of the Lomax distribution under G Type-I HCS.

In general, it has been observed that:

- The RMSE of $\hat{H}(x)$at α = 0.5, λ= 2.5 in Table 2, has the smallest value compared to the RMSE of $\hat{H}$ in the other Tables.
- In Tables 1, 2 and 3, for a fixed n, α, λ and r_1, the RMSE values of $\hat{H}(x)$ increase generally as the stopping time T increases, which usually results in a larger number of failures.
- In Tables 4, 5 and 6, for a fixed n, α, λ and T, the RMSE values of $\hat{H}(x)$ decrease generally as the number of failures r_2 increases, the same conclusion as in the previous item.
- For a fixed α, the RMSE values of $\hat{H}(x)$ decrease generally as the scale parameter λincreases.

In general, we observe that the RMSE values of $\hat{H}(x)$ decrease as the sample size n increase.

5. Summary

In this article, entropy estimates for the Lomax distribution were computed using the MLE of α and λ based on G Type-I HCS. The estimates were assessed in terms of their RMSE and it has been found the RMSEs are fairly small. Also, we obtained the approximate Fisher information matrix. To assess the performance of the estimates, we performed simulation studies with different sample sizes focusing on the entropy estimate of the Lomax distribution under the G Type-I HCS.

Appendix

Table 1: Bias estimates, relative mean square error (RMSE), entropy and relative MSE of entropy for α = 0.5 and λ = 3 by selected values n, r_1, r_2 and T

n	r_1	r_2	T	Bias α	Bias λ	RMSE α	RMSE λ	$\hat{H}$	RMSE $\hat{H}$
150	40	115	50	0.071	0.337	0.167	0.126	5.162	0.717
			60	0.083	0.337	0.200	0.126	5.256	0.088

			70	0.119	0.604	0.235	0.252	5.469	0.123
100	25	75	50	0.099	0.417	0.249	0.173	5.355	0.105
			60	0.269	0.519	0.298	0.209	5.459	0.122
			70	0.140	0.640	0.393	0.277	5.666	0.154
75	20	55	50	0.130	0.612	0.353	0.257	5.573	0.140
			60	0.142	0.608	0.401	0.279	5.687	0.157
			70	0.151	0.447	0.435	0.108	5.841	0.179

Table 2: Bias estimates, relative mean square error (RMSE), entropy and relative MSE of entropy for α = 0.5 and λ = 2.5 by selected values n, r_1, r_2 and T

n	r_1	r_2	T	Bias α	Bias λ	RMSE α	RMSE λ	$\widehat{H}$	RMSE $\widehat{H}$
150	40	115	50	0.060	0.138	0.138	0.058	5.058	0.088
			60	0.130	0.563	0.353	0.141	5.363	0.141
			70	0.108	0.453	0.275	0.221	5.204	0.114
100	25	75	50	0.091	0.252	0.224	0.112	5.154	0.105
			60	0.118	0.418	0.309	0.201	5.315	0.133
			70	0.138	0.570	0.382	0.295	5.438	0.152
75	20	55	50	0.127	0.127	0.343	0.207	5.404	0.147
			60	0.137	0.353	0.375	5.532	5.532	0.166
			70	0.137	0.353	0.377	0.164	5.532	0.166

Table 3: Bias estimates, relative mean square error (RMSE), entropy and relative MSE of entropy for α = 0.5 and λ = 1.5 by selected values n, r_1, r_2 and T

n	r_1	r_2	T	Bias α	Bias λ	RMSE α	RMSE λ	$\widehat{H}$	RMSE $\widehat{H}$
150	40	115	50	0.082	0.012	0.258	0.025	5.015	0.183
			60	0.136	0.110	0.374	0.079	5.089	0.194
			70	0.200	0.886	0.667	1.445	5.011	0.182
100	25	75	50	0.148	0.333	0.423	0.286	5.047	0.188
			60	0.174	0.548	0.534	0.576	5.141	0.203
			70	0.168	0.343	0.509	0.297	5.268	0.222
75	20	55	50	0.200	0.518	0.671	0.527	5.531	0.259
			60	0.208	0.627	0.719	0.719	5.521	0.257
			70	0.217	0.759	0.771	1.024	5.508	0.256

Table 4: Bias estimates, relative mean square error (RMSE), entropy and relative MSE of entropy for α = 0.5 and λ = 3 by selected values n, r_1, T and r_2

n	r_1	T	r_2	Bias α	Bias λ	RMSE α	RMSE λ	$\widehat{H}$	RMSE $\widehat{H}$
150	40	50	60	0.085	0.074	0.125	0.002	5.124	0.065
			75	0.100	0.254	0.152	0.829	5.225	0.829
			120	0.102	0.424	0.257	0.164	5.383	0.109
100	25		40	0.002	0.124	0.012	0.021	5.125	0.165
			50	0.015	0.015	0.025	0.036	5.012	0.044
			80	0.048	0.227	0.108	0.081	5.033	0.048
75	20		60	0.134	0.289	0.125	0.002	5.320	0.099

			35	0.251	0.258	0.025	0.015	5.054	0.052
			30	0.067	0.425	0.156	0.016	5.097	0.052

Table 5: Bias estimates, relative mean square error (RMSE), entropy and relative MSE of entropy for α = 0.5 and λ = 2.5 by selected values n, r_1, T and r_2

n	r_1	T	r_2	Bias α	Bias λ	RMSE α	RMSE λ	$\widehat{H}$	RMSE $\widehat{H}$
150	40	50	110	0.106	0.372	0.270	0.175	5.227	0.118
			75	0.154	0.485	0.299	0.201	5.362	0.140
			60	0.258	0.625	0.301	0.258	5.214	0.115
100	25		75	0.112	0.254	0.258	0.118	5.281	0.127
			50	0.201	0.335	0.315	0.239	5.236	0.119
			40	0.231	0.421	0.331	0.302	5.345	0.137
75	20		60	0.012	0.021	0.207	0.012	5.321	0.133
			35	0.458	0.145	0.632	0.190	5.342	0.137
			30	0.504	0.224	0.482	0.224	5.401	0.146

Table 6: Bias estimates, relative mean square error (RMSE), entropy and relative MSE of entropy for α = 0.5 and λ = 1.5 by selected values n, r_1, T and r_2

n	r_1	T	r_2	Bias α	Bias λ	RMSE α	RMSE λ	$\widehat{H}$	RMSE $\widehat{H}$
150	40	50	110	0.110	0.301	0.282	0.251	4.686	0.125
			75	0.251	0.578	0.362	0.485	4.558	0.100
			60	0.458	0.698	0.478	0.352	4.778	0.142
100	25		75	0.105	0.159	0.267	0.188	4.758	0.138

		50	0.258	0.015	0.235	0.258	4.658	0.12
		40	0.365	0.254	0.147	0.458	4.965	0.174
75	20	60	0.114	0.226	0.297	0.178	4.789	0.144
		35	0.198	0.244	0.402	0.201	4.825	0.150
		30	0.254	0.365	0.447	0.235	4.932	0.169

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