Handle outliers for Fréchet distribution

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Abstract

This paper aims to handle outlier data for Fréchet distribution. This study focused on two ways to deal with outliers. The first way is to censor the observation with the same percentage of outlier data. The second way is to trim outlier observations. A Monte Carlo simulation study is carried out to compare these ways in terms of estimate average, relative bias, and root mean square error (RMSE) using Mathematica-10.

Keywords— Outlier Data, Censored Data, Trimmed Data, Estimation, Fréchet Distribution

1 Introduction

The Fréchet Distribution was introduced by Fréchet (1924). It is used in many applications such as earthquakes, floods, precipitation, sea waves, supermarket queues, wind speed, and horse racing. The probability density function (pdf) of the Fréchet distribution takes the form:

$$f(x;\alpha,\beta)=\beta\alpha^{\beta}x^{-\beta-1}e^{-\left(\frac{\alpha}{x}\right)^{\beta}}, \tag{1.1}$$

where $\alpha>0$ is a scale parameter and $\beta>0$ is a shape parameter.
The cumulative distribution function (CDF) takes the form:

$$F(x;\alpha,\beta)=e^{-\left(\frac{\alpha}{x}\right)^{\beta}}, \tag{1.2}$$

and the corresponding quantile function is

$$q(u)=\alpha[-\log(u)]^{\frac{-1}{\beta}},\quad 0\le u\le 1. \tag{1.3}$$

For estimating the unknown parameters in the case of outliers Sillitto (1951) introduced the idea of linear moments, which was formally defined as expectations of certain linear combinations of order statistics by Hosking (1990).

The first way to deal with outliers is to censor data. Hosking (1995) defined two variants of L-moments which he used with right-censored data, while Zafirakou-Koulouris et al. (1998) extended the applicability of L-moments to left-censored data. Mahmoud et al. (2017) introduced the concept of Direct L-moments. They applied their method to the Kumaraswamy distribution and compared it with L-moments via the PPWM method and ML method.

The second way to deal with outliers is trimmed data. Elamir and Seheult [3] introduced an alternative robust approach of L-moments which they called trimmed L-moments (TL-moments). TL-moments have some advantages over L-moments and the method of moments. TL-moments exist whether or not the mean exists (for example, the Cauchy distribution), and they are more robust to the presence of outliers.

This work is organized as follows; Ways to handle the outliers are presented in Section 2. Outlier data from right for Fréchet distribution is presented in Section 3. Outlier data from left for Fréchet distribution is presented in Section 4. Simulation study and concluding remarks are presented in section 5 and 6 respectively.

2 Ways to handle the outliers

To handle the outlier data, this study focused on two ways to deal with outliers. The first way is to censor the observation with the same percentage of outlier data. The second way is to trim outlier observations.

2.1 Censored Data

Suppose that X is a real-valued random variable with cumulative distribution function $F(x)$ and quantile function $q(u)$, and let $X_{1:n} \leq X_{2:n} \leq \dots X_{n:n}$ be the order statistics of a random sample of size n drawn from the distribution of X. Hosking (1990) defined the L-moments of X to be the quantities

$$\lambda_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}E(X_{r-k:r}), \quad r = 1, 2, \dots \tag{2.1}$$

The L in 'L-moments' emphasizes that λ_r is a linear function of the expected order statistics. The expectation of an order statistic may be written as

$$E(X_{i:r}) = \frac{r!}{(i-1)!(r-i)!}\int_0^1 q(u)u^{i-1}(1-u)^{r-i}du. \tag{2.2}$$

Mahmoud et al. (2017) introduced two variants methods which are used with both right/left censored data. Let $x_1, x_2, ..., x_n$ be a Type-I censored random sample of size n from a distribution function $F(x)$ and quantile function $q(u)$. Let censoring time T satisfy $F(T) = c$, then c is the fraction of observed data.

$$\underbrace{x_{1:n} \leq x_{2:n} \leq \cdots \leq x_{m:n}}_{\text{m (observed)}} \leq T \leq \underbrace{x_{m+1:n} \leq \cdots \leq x_{n-1:n} \leq x_{n:n}}_{\text{n-m (censored)}}$$

Mahmoud et al. (2017) introduced two different Direct L-moments for using in L-moments with right-censored data, which they called Type-AD and Type-BD. The r^{th} population Type-AD Direct L-moments is:

$$\mu_r^A = \frac{(r-1)!}{c^r}\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k-1)!k!}\binom{r-1}{k}\int_0^c u^{r-k-1}(c-u)^k q(u)du. \tag{2.3}$$

Using the method of expectations, Type-AD L-moments estimators are given by:

$$M_r^A = \frac{1}{r\binom{m}{r}}\sum_{i=1}^{m}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r-k-1}\binom{m-i}{k}X_{i:n}. \tag{2.4}$$

The r^{th} population Type-BD Direct L-moments is:

$$\mu_r^B = (r-1)!\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k-1)!k!}\binom{r-1}{k} * \left[\beta^c(r-k,k+1)q(c) + \int_0^c u^{r-k-1}(1-u)^k q(u)du\right]. \tag{2.5}$$

Using the method of expectations, Type-BD L-moments estimators are given by:

$$M_r^B = \frac{1}{r\binom{n}{r}}\left[\sum_{i=1}^{m}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r-k-1}\binom{n-i}{k}X_{i:n} + \sum_{i=m+1}^{n}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r-k-1}\binom{n-i}{k}T\right]. \tag{2.6}$$

Let $x_1, x_2, ..., x_n$ be a random sample of size n. Type-I left censoring occurs when the observations below censoring time T are censored:

$$\underbrace{x_{1:n} \leq x_{2:n} \leq \cdots \leq x_{s:n}}_{\text{s (censored)}} \leq T \leq \underbrace{x_{s+1:n} \leq \cdots \leq x_{n-1:n} \leq x_{n:n}}_{\text{n-s (observed)}}$$

Let censoring time T satisfy $F(T) = h$, then h is the fraction of censored data. Mahmoud et al. (2017) introduced two different Direct L-moments for using in L-moments with left censored data, which they called Type-A'D and Type-B'D. The r^{th} population Type-A'D Direct L-moments is:

$$\mu_r^{A'} = \frac{(r-1)!}{(1-h)^r}\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k-1)!k!}\binom{r-1}{k}\int_h^1 (u-h)^{r-k-1}(1-u)^k q(u)du. \tag{2.7}$$

The r^{th} population Type-B'D Direct L-moments is:

$$\mu_r^{B'} = (r-1)!\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k-1)!k!}\binom{r-1}{k} * \left[\beta_h(r-k,k+1)q(h) + \int_h^1 u^{r-k-1}(1-u)^k q(u)du\right]. \tag{2.8}$$

Using the method of expectations, Type-A'D L-moments estimators are given by:

$$M_r^{A'} = \frac{1}{r\binom{n-s}{r}}\sum_{i=1}^{n-s}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r-k-1}\binom{n-s-i}{k}X_{s:n}, \tag{2.9}$$

where, $s = n - m + 1$. And, Type-B'D L-moments estimators are given by:

$$M_r^{B'} = \frac{1}{r\binom{n}{r}}\left[\sum_{i=1}^{s}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r-k-1}\binom{n-i}{k}T + \sum_{i=s+1}^{n}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r-k-1}\binom{n-i}{k}X_{i:n}\right]. \tag{2.10}$$

2.2 Trimming Data

? (?) introduced TL-moments. The idea of TL-moments is that the expected value $E(X_{r-k:r})$ is replaced with the expected value $E(X_{r+t_1-k:r+t_1+t_2})$. Thus, for each r, we increase the sample size of a random sample from the original r to $r+t_1+t_2$, working only with the expected values of these r modified order statistics $X_{t_1+1:r+t_1+t_2}, X_{t_1+2:r+t_1+t_2}, ..., X_{t_1+r:r+t_1+t_2}$ by trimming the smallest t_1 and the largest t_2 from the conceptual random sample.

The TL-moment of the r^{th} order of the random variable X is defined as:

$$\lambda_r^{(t_1,t_2)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}E(X_{r+t_1-k:r+t_1+t_2}), \quad r = 1, 2, \ldots. \tag{2.11}$$

The expectation of the order statistics is given by:

$$E(X_{i:r}) = \frac{r!}{(i-1)!(r-i)!}\int_0^1 q(u)u^{i-1}(1-u)^{r-i}du. \tag{2.12}$$

? (?) presented the following estimator for sample TL-moments:

$$l_r^{(t_1,t_2)} = \frac{1}{r\binom{n}{r+t_1+t_2}}\sum_{i=t_1+1}^{n-t_2}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r+t_1-k-1}\binom{n-i}{k+t_2}X_{i:n}. \tag{2.13}$$

Its basic idea for the method of expectation (Analogy principle) is to take the expected values of some functions of the random variable of interest and extend them to a sample and equate the corresponding results and solve for the unknown parameters.

3 Outlier data from right for Fréchet distribution

When you notice outlier data on the right, we compare between L-moments with Type-AD and Type-BD in the censored method, and also TL-moments with one largest value were trimmed. In this section, the r^{th} population L-moments and TL-moments for the Fréchet distribution are introduced.

3.1 Censored Data

From equations (1.3) and (2.3), the rth population Type-AD Direct L-moments for Type-I right censoring to the Fréchet distribution is:

$$\mu_r^A = \frac{\alpha(r-1)!}{c^r}\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k-1)!k!}\binom{r-1}{k}\int_0^c u^{r-k-1}(c-u)^k[-\log(u)]^{\frac{-1}{\beta}}du. \tag{3.1}$$

Substituting $r = 1, 2$ in equation (3.1); the first two Type-AD for Fréchet distribution will be:

$$\mu_1^A = \frac{\alpha}{c}\int_0^c[-\log(u)]^{\frac{-1}{\beta}}du.$$

Putting $z = -\log(u)$. Thus,

$$\mu_1^A = \frac{\alpha}{c}\int_{-\log(c)}^{\infty} z^{\frac{-1}{\beta}}e^{-z}dz.$$

Using the results in the appendix, μ_1^A can be written as:

$$\mu_1^A = \frac{\alpha}{c}\Gamma(\frac{-1}{\beta}+1,-\log(c)), \tag{3.2}$$

where, $\Gamma(c,b)$ is the upper incomplete gamma function.
Also,

$$\begin{aligned}\mu_2^A &= \frac{2\alpha}{c^2}\int_0^c u[-\log(u)]^{\frac{-1}{\beta}}du - \frac{\alpha}{c}\int_0^c [-\log(u)]^{\frac{-1}{\beta}}du \\ &= \frac{2\alpha}{c^2}\int_{-\log(c)}^{\infty} z^{\frac{-1}{\beta}}e^{-2z}dz - \frac{\alpha}{c}\int_{-\log(c)}^{\infty} z^{\frac{-1}{\beta}}e^{-z}dz.\end{aligned}$$

Using the results in the appendix, μ_2^A can be written as:

$$\mu_2^A = \frac{2^{\frac{1}{\beta}}\alpha}{c^2}\Gamma(\frac{-1}{\beta}+1,-2\log(c)) - \frac{\alpha}{c}\Gamma(\frac{-1}{\beta}+1,-\log(c)). \tag{3.3}$$

From equations (1.3) and (2.5), the rth population Type-BD Direct L-moments for Type-I right censoring to the Fréchet distribution is:

$$\begin{aligned}\mu_r^B = \alpha(r-1)!\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k-1)!k!}\binom{r-1}{k} \quad * \\ \left[\beta^c(r-k,k+1)(-\log(c))^{\frac{-1}{\beta}} + \int_0^c u^{r-k-1}(1-u)^k(-\log(u))^{\frac{-1}{\beta}}du\right].\end{aligned} \tag{3.4}$$

Substituting $r=1,2$ in equation (3.4); the first two Type-BD for Fréchet distribution will be:

$$\mu_1^B = \alpha(1-c)(-\log(c))^{\frac{-1}{\beta}} + \alpha\Gamma(\frac{-1}{\beta}+1,-\log(c)). \tag{3.5}$$

And,

$$\mu_2^B = \alpha c(1-c)(-\log(c))^{\frac{-1}{\beta}} - \alpha\Gamma(\frac{-1}{\beta}+1,-\log(c)) + 2^{\frac{1}{\beta}}\alpha\Gamma(\frac{-1}{\beta}+1,-2\log(c)). \tag{3.6}$$

3.2 Trimmed Data

From equation (2.11) the TL-moment of the r^{th} order of the random variable X with one largest value is trimmed is:

$$\lambda_r^{(0,1)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}E(X_{r-k:r+1}), \quad r=1,2,\ldots. \tag{3.7}$$

The expectation of the order statistics is given by:

$$E(X_{i:r}) = \frac{r!}{(i-1)!(r-i)!}\int_0^1 q(u)u^{i-1}(1-u)^{r-i}du.$$

And, the first four TL-moments with one largest value that were trimmed are:

$$\lambda_1^{(0,1)} = E(X_{1:1}), \tag{3.8a}$$

$$\lambda_2^{(0,1)} = \frac{1}{2}[E(X_{2:3}) - E(X_{1:3})], \tag{3.8b}$$

$$\lambda_3^{(0,1)} = \frac{1}{3}[E(X_{3:4}) - 2E(X_{2:4}) + E(X_{1:4})], \tag{3.8c}$$

$$\lambda_4^{(0,1)} = \frac{1}{4}[E(X_{4:5}) - 3E(X_{3:5}) + 3E(X_{2:5}) - E(X_{1:5})]. \tag{3.8d}$$

From equations (1.3) and (3.7), the rth population TL-moments with one largest value were trimmed to the Fréchet distribution is:

$$\lambda_r^{(0,1)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} E(X_{r-k:r+1}), \quad r = 1, 2, \ldots . \tag{3.9}$$

where the expectation of the order statistics is given by:

$$E(X_{i:r}) = \frac{r!\alpha}{(i-1)!(r-i)!}\int_0^1 [-\log(u)]^{\frac{-1}{\beta}} u^{i-1}(1-u)^{r-i} du. \tag{3.10}$$

From Equation (4.8a) and Equation (3.10), the first TL-moments with one largest value were trimmed to the Fréchet distribution is calculated as follows:

$$\lambda_1^{(0,1)} = \alpha \int_0^1 [-\log(u)]^{\frac{-1}{\beta}} du$$

Thus,

$$\lambda_1^{(0,1)} = \alpha\Gamma(\frac{-1}{\beta}+1) \tag{3.11}$$

From Equation (4.8b) and Equation (3.10), the second TL-moments with one largest value were trimmed to the Fréchet distribution is calculated as follows:

$$\begin{aligned}\lambda_2^{(0,1)} &= \frac{3\alpha}{2}\left[2\int_0^1 [-\log(u)]^{\frac{-1}{\beta}} u(1-u)du - \int_0^1 [-\log(u)]^{\frac{-1}{\beta}}(1-u)^2 du\right] \\ &= \frac{3\alpha}{2}\int_0^1 [-\log(u)]^{\frac{-1}{\beta}}(-1+4u-3u^2)du\end{aligned}$$

Thus,

$$\lambda_2^{(0,1)} = \frac{3\alpha}{2}(-1+2^{1+\frac{1}{\beta}} - 3^{\frac{1}{\beta}})\Gamma(\frac{-1}{\beta}+1) \tag{3.12}$$

4 Outlier data from left for Fréchet distribution

When you notice outlier data on the left, we compare between L-moments with Type-A'D and Type-B'D in the censored method, and also TL-moments with one smallest value were trimmed. In this section, the r^{th} population L-moments and TL-moments for the Fréchet distribution are introduced.

4.1 Censored Data

From equations (1.3) and (2.7), the rth population Type-A'D Direct L-moments for Type-I left censoring to the Fréchet distribution is:

$$\mu_r^{A'} = \frac{\alpha(r-1)!}{(1-h)^r}\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k-1)!k!}\binom{r-1}{k}\int_h^1 (u-h)^{r-k-1}(1-u)^k[-\log(u)]^{\frac{-1}{\beta}} du. \tag{4.1}$$

Substituting $r = 1, 2$ in equation (4.1); the first two Type-A'D for Fréchet distribution will be:

$$\mu_1^{A'} = \frac{\beta}{1-h}\int_h^1 [-\log(u)]^{\frac{-1}{\beta}} du.$$

Putting $z = -\log(u)$. Thus,

$$\mu_1^{A'} = \frac{\alpha}{1-h}\int_0^{-\log(h)} z^{\frac{-1}{\beta}} e^{-z} dz.$$

Using the results in the appendix, $\mu_1^{A'}$ can be written as:

$$\mu_1^{A'} = \frac{\alpha}{1-h}\gamma(\frac{-1}{\beta}+1, -\log(h)), \tag{4.2}$$

where, $\gamma(c, b)$ is the lower incomplete gamma function.
Also,

$$\mu_2^{A'} = \frac{\alpha}{(1-h)^2}\left(\int_h^1 (u-h)[-\log(u)]^{\frac{-1}{\beta}} du - \int_h^1 (1-u)[-\log(u)]^{\frac{-1}{\beta}} du\right).$$

Using the results in the appendix, $\mu_2^{A'}$ can be written as:

$$\mu_2^{A'} = \frac{\alpha}{(1-h)^2}\left[-(1+h)\gamma(\frac{-1}{\beta}+1, \log(h)) + 2^{\frac{1}{\beta}}\gamma(\frac{-1}{\beta}+1, -2\log(h))\right]. \tag{4.3}$$

From equations (1.3) and (2.8), the rth population Type-B'D Direct L-moments for Type-I left censoring to the Fréchet distribution is:

$$\mu_r^{B'} = \alpha(r-1)!\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k-1)!k!}\binom{r-1}{k} * \\ \left[\beta_h(r-k, k+1)[-\log(h)]^{\frac{-1}{\beta}} + \int_h^1 u^{r-k-1}(1-u)^k[-\log(u)]^{\frac{-1}{\beta}} du\right]. \tag{4.4}$$

Substituting $r = 1, 2$ in equation (4.4); the first two Type-B'D for Fréchet distribution will be:

$$\mu_1^{B'} = h\alpha(-\log(h))^{\frac{-1}{\beta}} + \alpha\gamma(\frac{-1}{\beta}+1, -\log(h)). \tag{4.5}$$

And,

$$\mu_2^{B'} = \alpha h(h-1)[-\log(h)]^{\frac{-1}{\beta}} - \alpha\gamma(\frac{-1}{\beta}+1, -\log(h)) + 2^{\frac{1}{\beta}}\alpha\gamma(\frac{-1}{\beta}+1, -2\log(h)). \tag{4.6}$$

4.2 Trimmed Data

From equation (2.11) the TL-moment of the r^{th} order of the random variable X with one smallest value is trimmed is:

$$\lambda_r^{(1,0)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}E(X_{r+1-k:r+1}), \quad r = 1, 2, \ldots. \tag{4.7}$$

The expectation of the order statistics is given by:

$$E(X_{i:r}) = \frac{r!}{(i-1)!(r-i)!}\int_0^1 q(u)u^{i-1}(1-u)^{r-i}du.$$

And, the first four TL-moments with one smallest value that were trimmed are:

$$\lambda_1^{(1,0)} = E(X_{2:2}), \tag{4.8a}$$

$$\lambda_2^{(1,0)} = \frac{1}{2}[E(X_{3:3}) - E(X_{2:3})], \tag{4.8b}$$

$$\lambda_3^{(1,0)} = \frac{1}{3}[E(X_{4:4}) - 2E(X_{3:4}) + E(X_{2:4})], \tag{4.8c}$$

$$\lambda_4^{(1,0)} = \frac{1}{4}[E(X_{5:5}) - 3E(X_{4:5}) + 3E(X_{3:5}) - E(X_{2:5})]. \tag{4.8d}$$

From equations (1.3) and (4.7), the rth population TL-moments with one smallest value were trimmed to the Fréchet distribution is:

$$\lambda_r^{(1,0)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} E(X_{r+1-k:r+1}), \quad r = 1, 2, \ldots. \tag{4.9}$$

where the expectation of the order statistics is given by:

$$E(X_{i:r}) = \frac{r!\alpha}{(i-1)!(r-i)!}\int_0^1 [-\log(u)]^{\frac{-1}{\beta}} u^{i-1}(1-u)^{r-i}du. \tag{4.10}$$

From Equation (4.8a) and Equation (4.10), the first TL-moments with one smallest value were trimmed to the Fréchet distribution is calculated as follows:

$$\lambda_1^{(1,0)} = 2\alpha \int_0^1 [-\log(u)]^{\frac{-1}{\beta}} du$$

Thus,

$$\lambda_1^{(1,0)} = 2^{\frac{1}{\beta}}\alpha\Gamma(\frac{-1}{\beta} + 1) \tag{4.11}$$

From Equation (4.8b) and Equation (4.10), the second TL-moments with one smallest value were trimmed to the Fréchet distribution is calculated as follows:

$$\begin{aligned}\lambda_2^{(1,0)} &= \frac{1}{2}\left[3\alpha\int_0^1[-\log(u)]^{\frac{-1}{\beta}}u^2du - 6\alpha\int_0^1[-\log(u)]^{\frac{-1}{\beta}}u(1-u)du\right] \\ &= \frac{3\alpha}{2}\int_0^1[-\log(u)]^{\frac{-1}{\beta}}(-2u+3u^2)du\end{aligned}$$

Thus,

$$\lambda_2^{(1,0)} = \frac{3\alpha}{2}(-2^{\frac{1}{\beta}} + 3^{\frac{1}{\beta}})\Gamma(\frac{-1}{\beta} + 1) \tag{4.12}$$

5 Simulation Study

This section is devoted to comparing three methods (*Direct L-moments*, L-moments, and maximum likelihood (ML)) in the estimation process using a comparative numerical study. We will estimate the two unknown parameters of the Fréchet distribution using these three methods given both right and left Type-I censored data. A comparative numerical study was carried out among the three methods based on estimate average, the root of mean square error (RMSE), and relative bias (RB). All computations are performed using Mathematica-10 programs. Following are the steps of this numerical study:

1. Generate random sample size n (20, 50 and 100) from the Fréchet distribution with parameters (α, β) take these initial values $(0.5, 5)$, $(2, 4)$ and $(0.2, 0.8)$.

2. The generated data is ordered.

3. Determine the level of censoring, take $c = 10\%$ and $c = 30\%$.

4. The simulation process to be repeated 5000 times.

5. Calculate means, the root of mean square error (RMSE), and relative biases (RB) for each sample size used and parameter values considered.

6. The simulation results are reported in Table(*) to Table (**).

6 Conclusion

The simulations show, Direct L-moments can be competitive with computationally more complex methods such as L-moments via PPWM and ML.

APPENDIX

The formal definition of the gamma function takes the following form:

$$\alpha^{-b}\Gamma(b) = \int_0^\infty x^{b-1}e^{-\alpha x}dx; \qquad \alpha > 0, \tag{A. 1}$$

putting $y = e^{-\alpha x}$, this form becomes:

$$\int_0^1 (-\log(y))^{b-1}dy = \Gamma(b). \tag{A. 2}$$

Additionally; the lower incomplete gamma function is:

$$\alpha^{-b}\gamma(\alpha c, b) = \int_0^c x^{b-1}e^{-\alpha x}dx, \tag{A. 3}$$

and it can be shown that:

$$\int_{e^{-c}}^1 (-\log(y))^{b-1}dy = \gamma(c, b). \tag{A. 4}$$

The upper incomplete gamma function is:

$$\alpha^{-b}\Gamma(\alpha c, b) = \int_c^\infty x^{b-1}e^{-\alpha x}dx, \tag{A. 5}$$

and it can be shown that:

$$\int_0^{e^{-c}} (-\log(y))^{b-1}dy = \Gamma(c, b). \tag{A. 6}$$

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