On Edge Covering Chain of a Graph

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Abstract- **In the study of domination in graphs, relationships between the concepts of maximal independent sets, minimal dominating sets and maximal irredundant sets are used to establish what is known as domination chain of parameters.**

$$ir(G) \le \gamma(G) \le i(G) \le \beta_0(G) \le \Gamma(G) \le IR(G)$$

In this paper,starting from the concept of edge cover, six graph theoretic parameters are introduced which obey a chain of inequalities,called as the edge covering chain of the graph G

Key words- **vertex cover, enclave less set, co-irredundant set, edge cover, edge enclave less set, edge co-irredundant set**

I. INTRODUCTION

By a graph $G = (V, E)$ we mean a finite,undirected graph with neither loops nor multiple edges.The order and size of G are denoted by n and m respectively.For graph theoretic terminology we refer to Chartrand and Lesniak[1]. Since this paper considers a relatively large number of graph parameters and inequlities between them we start with the following definitions and theorems.

1. $\alpha(G)$,the vertex covering number equals the minimum number of vertices in a vertex cover,that is a set $S \subseteq V$ having the property that for every edge $uv \in E$, either $u \in S$ or $v \in S$

2. $\Lambda(G)$,the upper vertex covering number equals the maximum number of vertices in a minimal vertex cover of G

3. $\beta_0(G)$,the vertex independence number equals the maximum number of vertices in an independent set,that is a set of vertices, no two of which are adjacent.

4. $i(G)$,the lower independence number equals the minimum number of vertices in a maximal independent set

5. $\gamma(G)$, the domination number equals the minimum number of vertices in a dominating set,that is a set $S \subseteq V$ for which every vertex in V-S is adjacent to atleast one vertex in S

6. $\Gamma(G)$,the upper domination number equals the maximum number of vertices in a minimal dominating set.

7. [5] $\Psi(G)$,the upper enclaveless number equals the maximum number of vertices in a set S, suchthat S has no enclave,that is a vertex $v \in S$ such that $N[v] \subseteq S$

8. $\psi(G)$,the lower enclaveless number equals the minimum number of vertices in a maximal enclaveless set S

9. *ir(G)*,the irredundance number equals the minimum number of vertices in a maximal irredundant set in G ,that is a set $S \subseteq V$ suchthat every vertex $v \in S$ has a private neighbour.

10. *IR(G)*,the upper irredundance number equals the maximum number of vertices in an irredundant set

11. *cir(G)*,the co- irredundance number equals the minimum number of vertices in a co-irredundant set in G,that is V-Sis irredundant

12. *CIR(G)*,equals the maximum number of vertices in a minimal co-irredundant in G

Theorem1.1[2] For any graph G,

$$ir(G) \le \gamma(G) \le i(G) \le \beta_0(G) \le \Gamma(G) \le IR(G)$$

Cockayneet.al [3] completely characterized the domination chain

Theorem1.2 [4] For any graph G,

$$cir(G) \le \psi(G) \le \alpha(G) \le \Lambda(G) \le \Psi(G) \le CIR(G)$$

Which is known as the Covering chain of a Graph G. Arumugam et al [4] characterized the covering chain and proved that it is the dual of domination chain

II. The Edge covering chain of the graph G

Definition2.1: The subset T of E is an edgecover if every vertex is incident to atleast one edge in T. T is a minimal edge cover if it has no proper subset which is an edge cover.

We denote $\alpha^{'}(G)$ and $\beta^{'}(G)$ by the minimum cardinality and maximum cardinality of minimal edge cover and call them as edge covering numbers

Definition2.2: An edge $e = uv \in T$ is an edge enclave if $N[e] \subseteq T$ where N[e]=N[u]$\cup$N[v]

A subset $T \subseteq E$ is an edge enclaveless set if every edge in T is adjacent to some edge in $E - T$. T is called maximal edge enclaveless set if it is not a proper subset of some other edge enclaveless set.

We denote $\psi^{'}(G)$ and $\Psi^{'}(G)$ by minimum cardinality and maximum cardinality of a maximal edge enclaveless set of G and call them as enclaveless numbersof G.

Clearly the property of being an edge cover is superhereditary property and an edge cover T is minimal if and only if T is l – minimal.

Preposition2.3: An edge cover T of E is a minimal edge cover if and only if T is an edge cover and is edge enclaveless.

Proof: Let T be a minimal edge cover of G and let $e \in T$ Then T – {e} is not an edge cover of G and hence $N[e] \not\subset T$ and T has no edge enclave

Conversely suppose T is an edge cover and edge enclaveless set. Suppose T is not a minimal edge cover, then these exists an edge $e \in T$ such that T –{e} is an edge cover. Then ‘e’ is an edge enclave in T which is a contradiction. Thus T is a minimal edge cover of G.

Proposition2.4: Every minimal edge cover T is a graph G is a maximal edge enclaveless set of G.

Proof: Let T be a minimal edge cover of G. Then by the above proposition, T is an edge enclaveless set.

If T is not a maximal enclaveless set, then these exists an edge $e \in E - T$ such that T U{e} is edge enclaveless. Hence these exists an edge $e_1 \in N(e) \cap (E-T)$ and the edges e and e1are adjacent in E – T which is a contradiction.Thus T is a maximal edge enclaveless set of G.

Corollary2.5: For any graph G $\psi'(G) \le \alpha'(G) \le \beta'(G) \le \Psi'(G)$

We observe that the property of being an edge enclavelss set is hereditary. Hence an edge enclavelss set T is maximal if and only if T is l – maximal.

Definition2.6: A set $T \subseteq E$ is called edge co-irredundant if E-T is edge irreundant ,that is every edge in E-T has edge private neighbour. The edge co-irredundance number cir'(G) and the upperedge co-irredundance number CIR'(G) are defined by

$cir'G) = \min\{|T|: \text{T is an edge co-irredundant set in G}\}$

$CIR'(G)=\max\{|T|: \text{T is a minimal edge co-irredundant set in G}\}$

The property of being egde co-irredundant is super hereditary.Hence an edge co-irredubdant set is minimal if and only if T is 1-minimal

Proposition 2.7: An edge enclaveless set T is a maximal enclaveless set if and only if T is edge enclaveless and V-T is edge irredundant.

Proof: Let T be a maximal enclavelessset.Then for any $e \in E-T, T \cup \{e\}$ contains an edge enclave e_1. Hence $N[e_1] \subseteq T \cup \{e\}$ Then $e \in E-T$ has a private neighbor with respect to *E-T.* Thus E – T is edge irredundant.

Conversely assume that T is edge enclaveless and E-T is edge irredundant. Then for any $e \in E-T$ there exists $e_1 \in N[e]$ suchthat $N[e_1] \cap (E-T) = \{e\}$.Then $e_1 \in T$ and e_1 is an edge enclave of $T \cup \{e\}$.Therefore T is a maximal edge enclaveless set.

Proposition 2.8: Every maximal edge enclaveless set T in a graph G is a minimal edge co-irredundant set G.

Proof: It follows from above proposition that *E – T* is edge irredundant and T is edge Co-irredundant.

If T is not a minimal edge co-irredundant set, then these exists $e \in T$ suchthat $T - \{e\}$ is an edge co-irredundant set. Hence $(E-T) \cup \{e\}$ is an edge – irredundant of G. Let e_1 be a edge private neighbor of 'e' with respect to $(E-T) \cup \{e\}$.Clearly $e_1 \in T$ and $N[e_1] \subseteq T$. Thus e_1 is an edge enclave in T which is a contadiction

Corollary2.9: For any graph G, We have

$$cir'(G) \le \psi'(G) \le \alpha'(G) \le \Lambda'(G) \le \Psi'(G) \le CIR'(G)$$

which is known as the Edge covering chain of a graph G

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