Numerical Solution of Fuzzy Delay Differential Equations by Fifth Order Runge-Kutta Method

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Abstract :

In this paper, we develop the numerical solutions of certain type called Fuzzy Delay Differential Equations(FDE) by using fifth order Runge-Kutta method for fuzzy differential equations. This method based on the seikkala derivative and finally we discuss the numerical examples to illustrate the theory.

Keywords : Fuzzy Differential Equations; Fuzzy Delay Differential Equations; Runge-Kutta Method of order five.

AMS Classification:65XXX

1. Introduction

The Delay differential equations are considered as a branch of ordinary differential equations its arise to describe the same physical phenomena, but they are different. The delay diffferential equations is the derivatives of unknown functions are dependent on the values of the functions at previous time. The concept of fuzzy derivative was first introduced by Chang, Zadeh in [7] it was followed up by Dubois, Prede in [9], who defined and used the extension principle. The study of fuzzy differential equations has been growing in recent years and has may application in science and engineering. The numerical method for solving fuzzy differential equations is introduced by Ma, Friedmen, Kandedl in [16] by the standard Euler method and by authors in [1, 2] by Taylor method. In the last few years many works have been performed by several authors in numerical solutions of fuzzy differential equations [1, 2, 3, 4, 5, 6]. D.Prasantha Bharathi and T.Jayakumar discussed Numerical Solution of fuzzy pure multiple Neutral Delay Differential Equations using Runge Kutta method [17]. Alfredo Bellan and Marino Zennaro studied numerical methods for delay differential equations in detail. D.Prasantha Bharathi et.al studied Exixtence and uniqueness of solution for Fuzzy Mixed type of Delay differential equations [18]. D.Prasantha Bhrathi and Jayakumar investigate different type of fuzzy Delay differential Equations with examples [14, 15, 17, 18, 19]. Also D.Prasantha Bharathi et.al discussed numerical solution of fuzzy multiple hybrid single neutral delay differential equations [20] .

Abbasbandy and Allahviranloo [3] discussed a numerical method for solving fuzzy differential equation by Runge-Kutta method of order four. Pederson and Sambandham [21] have investigated the numerical solution of hybrid fuzzy differential equation by using Runge-Kutta method. Al-Rawi et all [23] have discussed a numerical method for solving Delay differential equations by Runge-Kutta method of order four.

In this article, we develop numerical method for addressing fuzzy delay differential equation by an application of the Runge-Kutta method of order four [23]. In Section 2 we discuss about the Fuzzy Delay Differential Equations(FDDE's). In Section 3 the R-K method of order five for approaching fuzzy delay differential equations is discussed. Section 4 contains numerical examples to illustrate the theory.

2. Fuzzy Delay Differential Equations (FDDE)

Let us consider the FDDE

$$\begin{cases} y'(t) = f(t, y(t), y(t-\tau)), \quad t \geq 0 \\ y(t) = \phi(t), \quad -\tau \leq t \leq 0 \\ y(t_0) = y_0 \in \phi(t) \end{cases} \tag{1}$$

where $f : [0, \infty) \times R \times R \to E^n$ and $\phi \in R$ is a continuous fuzzy mapping and the initial condition $y_0 \in \phi$ then $y_0(s) = y(s) = \phi(s), \quad -\tau \leq s \leq 0$. Also y_0 is a fuzzy number with α-level intervals $[y_0]^\alpha = [\underline{y}_0^\alpha, \overline{y}_0^\alpha], \quad 0 \leq \alpha \leq 1$. The extension principle of Zadeh leads to the following definition of $g(t, y(t), y(t-\tau))$ when y is a fuzzy number.

It follows that

$$g(t,y(t),y(t-\tau)) = \begin{cases} \min g(t,u(t),v(t-\tau)) : u(t) \in (\underline{y}(t)^\alpha, \overline{y}(t)^\alpha), v(t-\tau) \in (\underline{y}(t-\tau)^\alpha, \overline{y}(t-\tau)^\alpha), \\ \max g(t,u(t),v(t-\tau) : u(t) \in (\underline{y}(t)^\alpha, \overline{y}(t)^\alpha), v(t-\tau) \in (\underline{y}(t-\tau)^\alpha, \overline{y}(t-\tau)^\alpha), \end{cases} \tag{2}$$

for $y \in E$ with α- level sets $[y]_\alpha = [\underline{y}^\alpha, \overline{y}^\alpha],\ 0 < \alpha \leq 1$

Since the fuzzy derivative $y'(t)$ of a fuzzy process, $y : R_+ \to E$ is defined by $[y'(t)]_\alpha = [(\underline{y}^\alpha)'(t), (\overline{y}^\alpha)'(t)]$, $0 < \alpha \leq 1$.

We call $y : R_+ \to E$ a fuzzy solution of (5) on the interval $I = [0, T]$ if

$$\begin{cases} (\underline{y}^\alpha)'(t) = \min f(t,u(t),v(t-\tau)) : u(t) \in (\underline{y}(t)^\alpha, \overline{y}(t)^\alpha), v(t-\tau) \in (\underline{y}(t-\tau)^\alpha, \overline{y}(t-\tau)^\alpha), \\ (\overline{y}^\alpha)'(t) = \max f(t,u(t),v(t-\tau) : u(t) \in (\underline{y}(t)^\alpha, \overline{y}(t)^\alpha), v(t-\tau) \in (\underline{y}(t-\tau)^\alpha, \overline{y}(t-\tau)^\alpha), \end{cases} \tag{3}$$

for $t \in I$ and $0 < \alpha \leq 1$.

Definition 2.1 [8]

Let I be a real interval and $F : I \to E^n$. If, for arbitrary fixed $t_0 \in I$ and $\epsilon > 0$, there exist $\delta > 0$, (depending on t_0 and ϵ) such that

$$t \in I, \quad |t - t_0| < \delta \quad \Rightarrow D(F(t), F(t_0)) < \epsilon,$$

then F is said to be continuous on I.

If $J = [a, b]$ is compact a compact interval in E, then $C(J, E^n)$ represents the set of all continuous fuzzy functions from J into E^n. In the space $C(J, E^n)$, we consider the following metric:

$$D(u, v) = \sup_{t \in J} D[u(t), v(t)].$$

Following the notation in [?], for a positive number τ, we denote by C_τ, the space $C([-\tau, 0], E^n)$, equipped with the metric defined by

$$D_\tau(u, v) = \sup_{t \in [-\tau, 0]} D[u(t), v(t)].$$

Remaining faithful to the classical notation used in the field of functional differential equations [10], for a given $u \in C([-\tau, \infty], E^n)$, u_t denotes for each $t \in [0, \infty)$, the element in C_τ, defined by

$$u_t(s) = u(t + s),\ s \in [-\tau, 0].$$

Lemma 2.1

If $g : [0, \infty) \times R \times R \to E^n$ is a jointly continuous function and $u : [-\tau, \infty) \to E^n$ is a continuous function, then the function

$$t \in [0, \infty) \to F(t, u(t), v(t-\tau)) \in E^n$$

is also continuous.

Theorem 2.1

Let $g : [0, \infty) \times R \times R \to E^n$ be a continuous fuzzy function such that there exists K and $M > 0$ such that $|(f(t, u_1, v_1) - f(t, u_2, v_2))| \leq K|u_1 - u_2| + M|v_1 - v_2|$, for all $t \in [0, \infty)$, $u_1, u_2, v_1, v_2 \in E^n$. Then (5) has a solution on I.

3. Fifth order Runge - Kutta Method

Here we consider for a FDDE's from equation (1),to constract the fuzzy delay differential equations (FDDE) via an application of Runge-Kutta method for fuzzy differential equation [3]using the method of Runge-Kutta method of order five when g in (1) can be obtained via the Zadeh extension principle from $g \in C[R^+ \times R \times R, R]$. We assume that the existence and uniqueness of solutions of (1) hold for each $[t_k, t_{k+1}]$.

This Runge - Kutta method is the fifth order approximation of $\underline{Y}'_k(t; \alpha)$ and $\overline{Y}'_k(t; \alpha)$.

We define

$$\underline{X}(t_{n+1}; \alpha) - \underline{X}(t_n; \alpha) = \sum_{i=1}^{6} w_i \underline{K}_i(t_n; x(t_n; \alpha)),$$

$$\overline{X}(t_{n+1}; \alpha) - \overline{X}(t_n; \alpha) = \sum_{i=1}^{6} w_i \overline{K}_i(t_n; x(t_n; \alpha)),$$

where w_1, w_2, w_3, w_5, w_5 and w_6 are constants and

$$\underline{K}_1(t; x(t; \alpha)) = \min\left\{ hg\Big(t, u(t), v(t-\tau)\Big) \Big| \; u(t) \in [\underline{x}(t; \alpha), \overline{x}(t; \alpha)], \; v(t-\tau) \in [\underline{x}(t-\tau; \alpha), \overline{x}(t-\tau; \alpha)] \right\},$$

$$\overline{K}_1(t; x(t; \alpha)) = \max\left\{ hg\Big(t, u(t), v(t-\tau)\Big) \Big| \; u(t) \in [\underline{x}(t; \alpha), \overline{x}(t; \alpha)], \; v(t-\tau) \in [\underline{x}(t-\tau; \alpha), \overline{x}(t-\tau; \alpha)] \right\},$$

$$\underline{K}_2(t; x(t; \alpha)) = \min\left\{ hg\Big(t + \frac{h}{2}, u(t), v(t-\tau)\Big) \Big| \; u(t) \in [\underline{z}_1(t, x(t; \alpha)), \overline{z}_1(t, x(t; \alpha))], \right.$$
$$\left. v(t-\tau) \in [\underline{z}_1(t-\tau, x(t-\tau; \alpha)), \overline{z}_1(t-\tau, x(t-\tau; \alpha))] \right\},$$

$$\overline{K}_2(t; x(t; \alpha)) = \max\left\{ hg\Big(t + \frac{h}{2}, u(t), v(t-\tau)\Big) \Big| \; u(t) \in [\underline{z}_1(t, x(t; \alpha)), \overline{z}_1(t, x(t; \alpha))], \right.$$
$$\left. v(t-\tau) \in [\underline{z}_1(t-\tau, x(t-\tau; \alpha)), \overline{z}_1(t-\tau, x(t-\tau; \alpha))] \right\},$$

$$\underline{K}_3(t; x(t; \alpha)) = \min\left\{ hg\Big(t + \frac{h}{4}, u(t), v(t-\tau)\Big) \Big| \; u(t) \in [\underline{z}_2(t, x(t; \alpha)), \overline{z}_2(t, x(t; \alpha))], \right.$$
$$\left. v(t-\tau) \in [\underline{z}_2(t-\tau, x(t-\tau; \alpha)), \overline{z}_2(t-\tau, x(t-\tau; \alpha))] \right\},$$

$$\overline{K}_3(t; x(t; \alpha)) = \max\left\{ hg\Big(t + \frac{h}{4}, u(t), v(t-\tau)\Big) \Big| \; u(t) \in [\underline{z}_2(t, x(t; \alpha)), \overline{z}_2(t, x(t; \alpha))], \right.$$
$$\left. v(t-\tau) \in [\underline{z}_2(t-\tau, x(t-\tau; \alpha)), \overline{z}_2(t-\tau, x(t-\tau; \alpha))] \right\},$$

$$\underline{K}_4(t; x(t; \alpha)) = \min\left\{ hg\Big(t + \frac{h}{2}, u(t), v(t-\tau)\Big) \Big| \; u(t) \in [\underline{z}_3(t, x(t; \alpha)), \overline{z}_3(t, x(t; \alpha))], \right.$$
$$\left. v(t-\tau) \in [\underline{z}_3(t-\tau, x(t-\tau; \alpha)), \overline{z}_3(t-\tau, x(t-\tau; \alpha))] \right\},$$

$$\overline{K}_4(t;x(t;\alpha)) = \max\left\{hg\left(t+\frac{h}{2},u(t),v(t-\tau)\right)\middle|\ u(t)\in[\underline{z}_3(t,x(t;\alpha)),\overline{z}_3(t,x(t;\alpha))],\right.$$
$$\left.v(t-\tau)\in[\underline{z}_3(t-\tau,x(t-\tau;\alpha)),\overline{z}_3(t-\tau,x(t-\tau;\alpha))]\right\},$$
$$\underline{K}_5(t;x(t;\alpha)) = \min\left\{hg\left(t+\frac{3h}{2},u(t),v(t-\tau)\right)\middle|\ u(t)\in[\underline{z}_4(t,x(t;\alpha)),\overline{z}_4(t,x(t;\alpha))],\right.$$
$$\left.v(t-\tau)\in[\underline{z}_4(t-\tau,x(t-\tau;\alpha)),\overline{z}_4(t-\tau,x(t-\tau;\alpha))]\right\},$$
$$\overline{K}_5(t;x(t;\alpha)) = \max\left\{hg\left(t+\frac{3h}{2},u(t),v(t-\tau)\right)\middle|\ u(t)\in[\underline{z}_4(t,x(t;\alpha)),\overline{z}_4(t,x(t;\alpha))],\right.$$
$$\left.v(t-\tau)\in[\underline{z}_4(t-\tau,x(t-\tau;\alpha)),\overline{z}_4(t-\tau,x(t-\tau;\alpha))]\right\},$$
$$\underline{K}_6(t;x(t;\alpha)) = \min\left\{hg\left(t+h,u(t),v(t-\tau)\right)\middle|\ u(t)\in[\underline{z}_5(t,x(t;\alpha)),\overline{z}_5(t,x(t;\alpha))],\right.$$
$$\left.v(t-\tau)\in[\underline{z}_5(t-\tau,x(t-\tau;\alpha)),\overline{z}_5(t-\tau,x(t-\tau;\alpha))]\right\},$$
$$\overline{K}_6(t;x(t;\alpha)) = \max\left\{hg\left(t+h,u(t),v(t-\tau)\right)\middle|\ u(t)\in[\underline{z}_5(t,x(t;\alpha)),\overline{z}_5(t,x(t;\alpha))],\right.$$
$$\left.v(t-\tau)\in[\underline{z}_5(t-\tau,x(t-\tau;\alpha)),\overline{z}_5(t-\tau,x(t-\tau;\alpha))]\right\},$$

Next we define

$$\underline{z}_1(t,x(t;\alpha)) = \underline{x}(t;\alpha)+\frac{1}{2}\underline{K}_1(t,x(t;\alpha)),$$
$$\overline{z}_1(t,x(t;\alpha)) = \overline{x}(t;\alpha)+\frac{1}{2}\overline{K}_1(t,x(t;\alpha)),$$
$$\underline{z}_2(t,x(t;\alpha)) = \underline{x}(t;\alpha)+\frac{3}{16}\underline{K}_2(t,x(t;\alpha))+\frac{1}{16}\underline{K}_2(t,x(t;\alpha)),$$
$$\overline{z}_2(t,x(t;\alpha)) = \overline{x}(t;\alpha)+\frac{3}{16}\overline{K}_1(t,x(t;\alpha))+\frac{1}{16}\overline{K}_2(t,x(t;\alpha)),$$
$$\underline{z}_3(t,x(t;\alpha)) = \underline{x}(t;\alpha)+\frac{1}{2}\underline{K}_3(t,x(t;\alpha)),$$
$$\overline{z}_3(t,x(t;\alpha)) = \overline{x}(t;\alpha)+\frac{1}{2}\overline{K}_3(t,x(t;\alpha)),$$
$$\underline{z}_4(t,x(t;\alpha)) = \underline{x}(t;\alpha)-\frac{3}{16}\underline{K}_2(t,x(t;\alpha))+\frac{6}{16}\underline{K}_3(t,x(t;\alpha))+\frac{9}{16}\underline{K}_3(t,x(t;\alpha)),$$
$$\overline{z}_4(t,x(t;\alpha)) = \overline{x}(t;\alpha)-\frac{3}{16}\overline{K}_2(t,x(t;\alpha))+\frac{6}{16}\overline{K}_3(t,x(t;\alpha))+\frac{9}{16}\overline{K}_4(t,x(t;\alpha)),$$
$$\underline{z}_5(t,x(t;\alpha)) = \underline{x}(t;\alpha)+\frac{1}{7}\underline{K}_1(t,x(t;\alpha))+\frac{4}{7}\underline{K}_2(t,x(t;\alpha))+\frac{6}{7}\underline{K}_3(t,x(t;\alpha))-\frac{12}{7}\underline{K}_4(t,x(t;\alpha))+\frac{8}{7}\underline{K}_5(t,x(t;\alpha)),$$
$$\overline{z}_5(t,x(t;\alpha)) = \overline{x}(t;\alpha)+\frac{1}{7}\overline{K}_1(t,x(t;\alpha))+\frac{4}{7}\overline{K}_2(t,x(t;\alpha))+\frac{6}{7}\overline{K}_3(t,x(t;\alpha))-\frac{12}{7}\overline{K}_4(t,x(t;\alpha))+\frac{8}{7}\overline{K}_5(t,x(t;\alpha)).$$

Next we define

$$S[(t,\underline{x}(t;\alpha),\overline{x}(t;\alpha))] = 7\underline{K}_1(t,x(t;\alpha))+32\underline{K}_3(t,x(t;\alpha))+12\underline{K}_4(t,x(t;\alpha))+32\underline{K}_5(t,x(t;\alpha))+7\underline{K}_6(t,x(t;\alpha)),$$
$$T[(t,\underline{x}(t;\alpha),\overline{x}(t;\alpha))] = 7\overline{K}_1(t,x(t;\alpha))+32\overline{K}_3(t,x(t;\alpha))+12\overline{K}_4(t,x(t;\alpha))+32\overline{K}_5(t,x(t;\alpha))+7\overline{K}_6(t,x(t;\alpha)).$$

The exact solution at t_{n+1} is given by

$$\begin{cases} \underline{X}(t_{n+1};\alpha) \approx \underline{X}(t_n;\alpha)+\dfrac{1}{90}S[(t_n,\underline{X}(t_n;\alpha),\overline{X}_n(t;\alpha))], \\ \overline{X}(t_{n+1};\alpha) \approx \overline{X}(t_n;\alpha)+\dfrac{1}{90}T[(t_n,\underline{X}(t_n;\alpha),\overline{X}_n(t;\alpha))]. \end{cases} \quad (4)$$

The approximate solution is given by

$$\begin{cases} \underline{x}(t_{n+1};\alpha) = \underline{x}(t_n;\alpha) + \dfrac{1}{90}S[(t_n, \underline{x}(t_n;\alpha), \overline{x}_n(t;\alpha))], \\ \overline{x}(t_{n+1};\alpha) = \overline{x}(t_n;\alpha) + \dfrac{1}{90}T[(t_n, \underline{x}(t_n;\alpha), \overline{x}_n(t;\alpha))]. \end{cases} \tag{5}$$

Theorem 3.1
Consider the systems (5) and (9). For a fixed $\alpha \in [0,1]$,

$$\lim_{h\to 0} \underline{X}(t_N;\alpha) = \underline{X}(t_N;\alpha),$$
$$\lim_{h\to 0} \overline{X}(t_N;\alpha) = \overline{X}(t_N;\alpha),$$

4. Numerical Examples
Consider the FDDE

$$\begin{cases} x'(t) = g(t, x(t), x(t-\tau)), & t \geq 0 \\ x(t) = \phi(t), & -\tau \leq t \leq 0 \end{cases} \tag{6}$$

where $\phi(t)$ be a initial function

$$[x(t)]^{\alpha} = [\underline{x}(t;\alpha), \overline{x}(t;\alpha)], \quad t \geq 0, \quad [\phi(t)]^{\alpha} = [\underline{\phi}(t;\alpha), \overline{\phi}(t;\alpha)], \quad t \in [-\tau, 0]$$

and

$$\begin{aligned} [g(t, x(t), x(t-\tau))]^{\alpha} = & \Big[\underline{g}(t, \underline{x}(t;\alpha), \overline{x}(t;\alpha), \underline{x}(t-\tau;\alpha), \overline{x}(t-\tau;\alpha)), \\ & \overline{g}(t, \underline{x}(t;\alpha), \overline{x}(t;\alpha), \underline{x}(t-\tau;\alpha), \overline{x}(t-\tau;\alpha))\Big], \quad t \geq 0 \end{aligned}$$

Example 4.1
Consider the FDDE

$$\begin{cases} x'(t) = x(t) \; + x(t-1), & t \geq 0, \\ x(t) = \phi(t), & -1 \leq t \leq 0 \end{cases} \tag{7}$$

Let $\phi(t) = [(0.75 + 0.25\alpha), (1.125 - 0.125\alpha)]$. $\alpha \in [0,1]$
By using fifth order Runge-Kutta method we have,
for $t \in [0,1]$

$$\begin{aligned} \underline{x}\Big(\frac{i}{10};\alpha\Big) &= \underline{x}\Big(\frac{i-1}{10};\alpha\Big)\Big(1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24} + \frac{h^5}{120} + \frac{h^6}{1280}\Big) \\ &\quad + \underline{x}\Big(\frac{i-1}{10} - 1;\alpha\Big)\Big(h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24} + \frac{h^5}{120} + \frac{h^6}{1280}\Big), \\ \overline{x}\Big(\frac{i}{10};\alpha\Big) &= \overline{x}\Big(\frac{i-1}{10};\alpha\Big)\Big(1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24} + \frac{h^5}{120} + \frac{h^6}{1280}\Big) \\ &\quad + \overline{x}\Big(\frac{i-1}{10} - 1;\alpha\Big)\Big(h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24} + \frac{h^5}{120} + \frac{h^6}{1280}\Big), \end{aligned}$$

where $i = 1, 2, \cdots 10$.
For $t \in [1,2]$

$$\underline{x}\Big(1 + \frac{i}{10};\alpha\Big) = \underline{x}\Big(1 + \frac{i-1}{10};\alpha\Big)\Big(1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24} + \frac{h^5}{120} + \frac{h^6}{1280}\Big)$$

$$
\begin{aligned}
& +\underline{x}\Big(\frac{i-1}{10};\alpha\Big)\Big(h+\frac{h^2}{2}+\frac{h^3}{6}+\frac{h^4}{24}+\frac{h^5}{120}+\frac{h^6}{1280}\Big)\\
& +\underline{x}\Big(\frac{i-1}{10};\alpha\Big)\Big(\frac{h^2}{2}+\frac{h^3}{3}+\frac{h^4}{8}+\frac{h^5}{30}+\frac{h^6}{256}\Big)+\underline{x}\Big(\frac{i-1}{10}-1;\alpha\Big)\Big(\frac{h^2}{2}+\frac{h^3}{3}+\frac{h^4}{8}+\frac{h^5}{30}+\frac{h^6}{256}\Big),\\
\overline{x}\Big(1+\frac{i}{10};\alpha\Big) = & \ \overline{y}\Big(1+\frac{i-1}{10};\alpha\Big)\Big(1+h+\frac{h^2}{2}+\frac{h^3}{6}+\frac{h^4}{24}+\frac{h^5}{120}+\frac{h^6}{1280}\Big)\\
& +\overline{x}\Big(\frac{i-1}{10};\alpha\Big)\Big(h+\frac{h^2}{2}+\frac{h^3}{6}+\frac{h^4}{24}+\frac{h^5}{120}+\frac{h^6}{1280}\Big)\\
& +\overline{x}\Big(\frac{i-1}{10};\alpha\Big)\Big(\frac{h^2}{2}+\frac{h^3}{3}+\frac{h^4}{8}+\frac{h^5}{30}+\frac{h^6}{256}\Big)+\overline{x}\Big(\frac{i-1}{10}-1;\alpha\Big)\Big(\frac{h^2}{2}+\frac{h^3}{3}+\frac{h^4}{8}+\frac{h^5}{30}+\frac{h^6}{256}\Big),
\end{aligned}
$$

where $i = 1, 2, \cdots 10$.
The exact solution of (11) is given by
$X(t;\alpha) = \Big[(0.75+0.25\alpha)(2e^t-1)\ ,\ (1.125-0.125\alpha)(2e^t-1)\Big]$, for $t \in [0,1]$,
$X(t;\alpha) = \Big[(0.75+0.25\alpha)(2(e^t-2e^{t-1}+te^{t-1})+1), (1.125-0.125\alpha)(2(e^t-2e^{t-1}+te^{t-1})+1)\Big]$
for $t \in [1,2]$.
The approximate solution for $t \in [0,2], \alpha \in [0,1]$, is shown in figure 2. The exact and approximate solution by fifth order Runge-Kutta method are compared and plotted at t=2 in figure 3 and the results of example 5.1 at t=2 are shown in table 1. The exact solution for $\alpha = 1$, $t \in [0,20]$ is shown in figure 4.

Table 1
Comparison of exact solution and approximate solution by fifth order Runge-Kutta method

α	R-K 5^{th} order		Exact Solution	
	$\underline{x}(t_i;\alpha)$	$\overline{x}(t_i;\alpha)$	$\underline{X}(t_i;\alpha)$	$\overline{X}(t_i;\alpha)$
0	11.8335517785545	17.7503276678317	11.8335841483960	17.7503762225940
0.1	12.2280035045063	17.5531018048558	12.2280369533425	17.5531498201207
0.2	12.6224552304581	17.3558759418799	12.6224897582890	17.3559234176474
0.3	13.0169069564099	17.1586500789040	13.0169425632356	17.1586970151742
0.4	13.4113586823617	16.9614242159281	13.4113953681821	16.9614706127009
0.5	13.8058104083136	16.7641983529522	13.8058481731286	16.7642442102276
0.6	14.2002621342654	16.5669724899763	14.2003009780752	16.5670178077544
0.7	14.5947138602172	16.3697466270004	14.5947537830217	16.3697914052811
0.8	14.9891655861690	16.1725207640245	14.9892065879682	16.1725650028078
0.9	15.3836173121208	15.9752949010485	15.3836593929148	15.9753386003346
1	15.7780690380726	15.7780690380726	15.7781121978613	15.7781121978613

Approximate solution by fifth order Runge-Kutta method

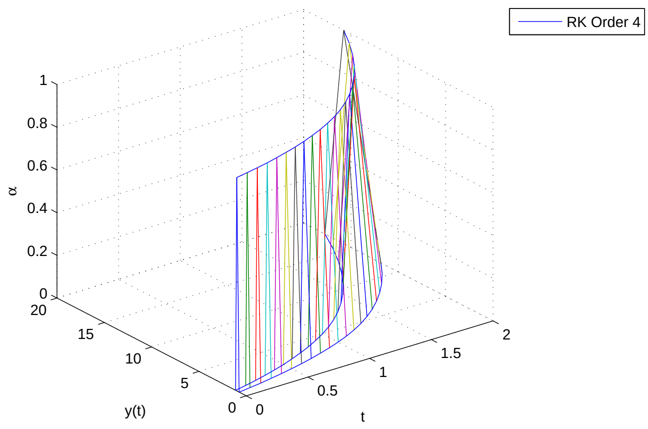

Figure 1: (for h=0.1)

Comparison of exact solution and approximate solution by fifth order R-K method

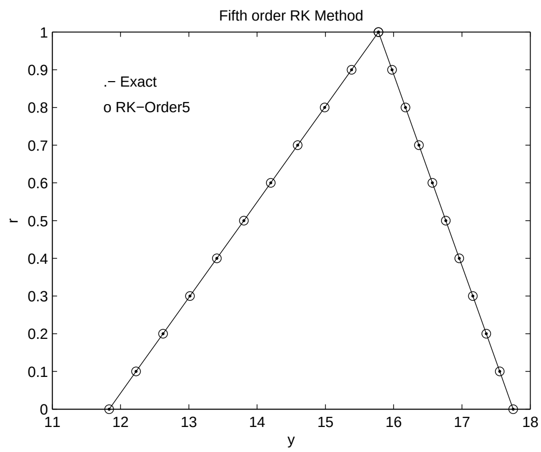

Figure 2: (for h=0.1 and t=2)

Exact solution for, $\alpha = 1$, $t \in [0, 20]$

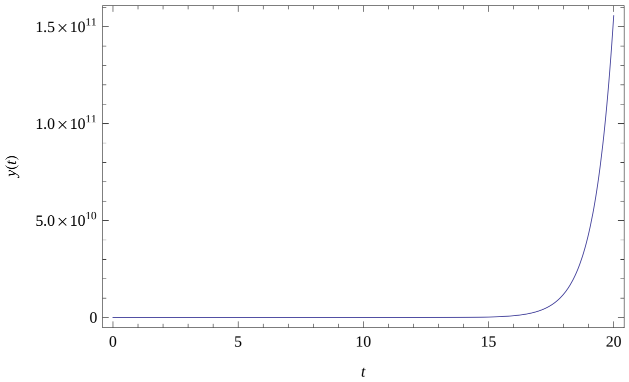

Figure 3:

Example 4.2

Consider the FDDE

$$\begin{cases} x'(t) = \lambda x(t-1), & t \geq 0, \\ x(t) = \phi(t), & -1 \leq t \leq 0 \end{cases} \tag{8}$$

Let $\phi(t) = [(0.75 + 0.25\alpha)e^t, (1.125 - 0.125\alpha)e^t]$.
The initial value is given by, $[x_0]^\alpha = [0.75 + 0.25\alpha, 1.125 - 0.125\alpha]$.
The exact solution of (12) is given by,

$$X(t;\alpha) = \Big[(0.75+0.25\alpha)(1+\frac{\lambda(e^t-1)}{e}), (1.125-0.125\alpha)(1+\frac{\lambda(e^t-1)}{e})\Big], \quad \text{for } t \in [0,1]$$

$$\begin{aligned} X(t;\alpha) &= \Big[(0.75+0.25\alpha)\Big(1+\lambda\Big(t+\lambda\frac{(e^{t-1}-t)}{e}+\frac{e-1}{e}-1\Big), \\ &\quad (1.125-0.125\alpha)\Big(1+\lambda\Big(t+\lambda\frac{(e^{t-1}-t)}{e}+\frac{e-1}{e}-1\Big)\Big]. \quad \text{for } t \in [1,2] \end{aligned}$$

When $\lambda = -1$ using fifth order Runge-Kutta method we have,
for $t \in [0,1]$

$$\underline{x}\Big(\frac{i}{10};\alpha\Big) = \underline{x}\Big(\frac{i-1}{10};\alpha\Big) - \underline{x}\Big(\frac{i-1}{10}-1;\alpha\Big)\Big(h+\frac{h^2}{2}+\frac{h^3}{6}+\frac{h^4}{24}+\frac{h^5}{120}+\frac{h^6}{1280}\Big),$$

$$\overline{x}\Big(\frac{i}{10};\alpha\Big) = \overline{x}\Big(\frac{i-1}{10};\alpha\Big) - \overline{x}\Big(\frac{i-1}{10}-1;\alpha\Big)\Big(h+\frac{h^2}{2}+\frac{h^3}{6}+\frac{h^4}{24}+\frac{h^5}{120}+\frac{h^6}{1280}\Big),$$

where $i = 1, 2, \cdots 10$
for $t \in [1,2]$

$$\underline{x}\Big(1+\frac{i}{10};\alpha\Big) = \underline{x}\Big(1+\frac{i-1}{10};\alpha\Big) - h\underline{x}\Big(\frac{i-1}{10};\alpha\Big) + \underline{x}\Big(\frac{i-1}{10}-1;\alpha\Big)\Big(\frac{h^2}{2}+\frac{h^3}{6}+\frac{h^4}{24}+\frac{h^5}{120}+\frac{h^6}{1280}\Big),$$

$$\overline{y}\Big(1+\frac{i}{10};\alpha\Big) = \overline{y}\Big(1+\frac{i-1}{10};\alpha\Big) - h\overline{x}\Big(\frac{i-1}{10};\alpha\Big) + \overline{x}\Big(\frac{i-1}{10}-1;\alpha\Big)\Big(\frac{h^2}{2}+\frac{h^3}{6}+\frac{h^4}{24}+\frac{h^5}{120}+\frac{h^6}{1280}\Big),$$

where $i = 1, 2, \cdots 10$
When $\lambda = -1$ the exact solution is given by,

$$X(t;\alpha) = \Big[(0.75+0.25\alpha)(1-\frac{(e^t-1)}{e}), (1.125-0.125\alpha)(1-\frac{(e^t-1)}{e})\Big], \quad \text{for } t \in [0,1]$$

$$\begin{aligned} X(t;\alpha) &= \Big[(0.75+0.25\alpha)\Big(1-\Big(t-\frac{(e^{t-1}-t)}{e}+\frac{e-1}{e}-1\Big), \\ &\quad (1.125-0.125\alpha)\Big(1-\Big(t-\frac{(e^{t-1}-t)}{e}+\frac{e-1}{e}-1\Big)\Big], \quad \text{for } t \in [1,2] \end{aligned}$$

The approximate solution for $t \in [0, 2], \alpha \in [0, 1]$, is shown in figure 5. The exact and approximate solution by fifth order Runge-Kutta method are compared and plotted at t=2 in figure 6 and the results of example 5.2 at t=2 are shown in table 2. The exact solution for $\alpha = 1$, $t \in [0, 20]$ is shown in figure 7.

Table 2
Comparison of exact solution and approximate solution by fifth order Runge-Kutta method

α	R-K 5^{th} order		Exact Solution	
	$\underline{x}(t_i;\alpha)$	$\overline{x}(t_i;\alpha)$	$\underline{X}(t_i;\alpha)$	$\overline{X}(t_i;\alpha)$
0	-0.275909721782563	-0.413864582673845	-0.275909580878582	-0.413864371317873
0.1	-0.285106712508649	-0.409266087310802	-0.285106566907868	-0.409265878303230
0.2	-0.294303703234734	-0.404667591947759	-0.294303552937154	-0.404667385288587
0.3	-0.303500693960820	-0.400069096584717	-0.303500538966440	-0.400068892273944
0.4	-0.312697684686905	-0.395470601221674	-0.312697524995726	-0.395470399259301
0.5	-0.321894675412990	-0.390872105858631	-0.321894511025012	-0.390871906244658
0.6	-0.331091666139076	-0.386273610495589	-0.331091497054298	-0.386273413230014
0.7	-0.340288656865161	-0.381675115132546	-0.340288483083584	-0.381674920215371
0.8	-0.349485647591247	-0.377076619769503	-0.349485469112870	-0.377076427200728
0.9	-0.358682638317332	-0.372478124406460	-0.358682455142156	-0.372477934186085
1	-0.367879629043418	-0.367879629043418	-0.367879441171442	-0.367879441171442

Approximate solution by fifth order Runge-Kutta method

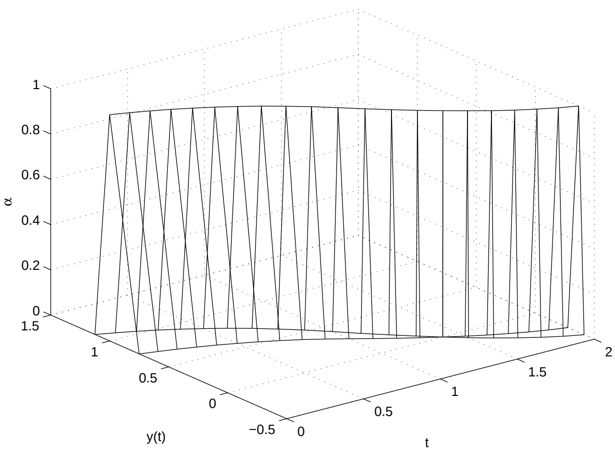

Figure 4: (for h=0.1)

Comparison of exact solution and approximate solution by fifth order R-K method

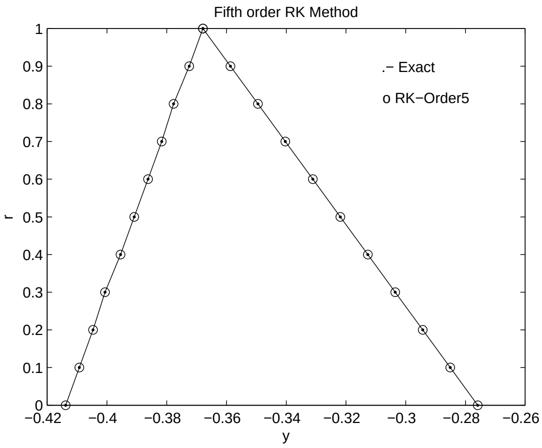

Figure 5: (for h=0.1 and t=1)

Exact solution for, $\alpha = 1$, $t \in [0, 20]$

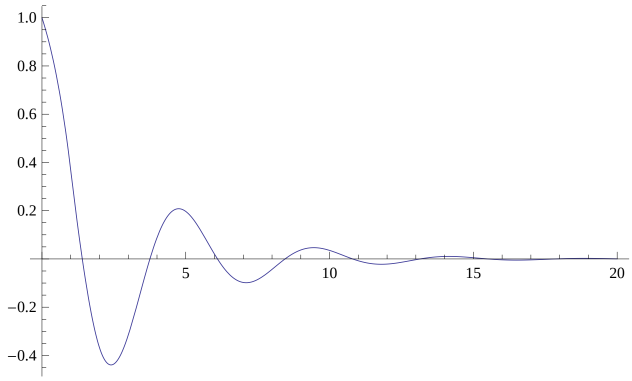

Figure 6:

5. Conclusion

In this paper, we presented a numerical iterative solution of fifth order Runge-Kutta method for finding the numerical solution of fuzzy delay differential equations based on Seikkala's derivative and Hukuhara differentiability of fuzzy process are considered. In the proposed method the convergence order is $O(h^5)$.

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