On Jordan ideals and Generalized (α, 1)- Reverse derivations in ∗-prime rings

Sk. Haseena[1]

[1]Research Scholar, Department of Mathematics,

S.V.University, Tirupati- 517502, A.P., India.

e-mail: skhaseena547@gmail.com

Chennupalle Divya[2]

[2]Department of CSE-ASEC, Amrita School of Engineering,

Bangalore, India.

e-mail: chennupalledivya@gmail.com

C. Jaya Subba Reddy[3]

[3]Department of Mathematics, S.V.University,

Tirupati- 517502, Andhra Pradesh, India.

e-mail: cjsreddysvu@gmail.com

ABSTRACT

Let R will be a 2- torsion free ∗-prime ring and α be an automorphisum of R. F be a nonzero generalized (α, 1)- reverse derivation of R with associated nonzero (α, 1)- reverse derivation d which commutes with ∗ and J be a nonzero ∗-Jordan ideal and a subring of R. In the present paper, we shall prove that R is commutative if any one of the following holds: (i) $[F(u), u]_{\alpha,1} = 0$, (ii) $F(u)\,\alpha(u) = ud(u)$, (iii) $F(u^2) = \pm\,\alpha(u^2)$, (iv) $F(u^2) = 2d(u)\,\alpha(u)$, (v) $d(u^2) = 2F(u)\,\alpha(u)$, for all $u \in U$.

Key words: ∗-prime ring, ∗-Jordan ideal, (α, 1)-reverse derivation, Generalized (α, 1)-reverse derivation.

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1. INTRODUCTION

The study of centralizing and commuting mappings was initiated by Posner [18]. Bresar and Vukman [4] have introduced the notion of a reverse derivation. Samman et.al. [19] and Ibraheem [12] have studied reverse derivations and generalized reverse derivation on semiprime or prime rings. Jaya Subba Reddy et.al. [8-11] have proved some results of generalized reverse derivations on prime rings, Centralizing and commuting left generalized derivations on Prime Rings, Homomorphism or Anti-homomorphism of left (α, 1)-derivations in Prime rings and (α,1) reverse derivations on prime near-rings. Oukhtite et.al.[15-17] have studied on generalized derivations of σ-prime rings, on Jordan ideals and derivations in rings with involution, Posner's second theorem for Jordan ideals in rings with involution. Several authors have proved commutativity theorems for prime rings or semiprime rings admitting auto-morphisms or derivations which are centralizing and commuting on appropriate subsets of R(See [2, 6, 13]), annihilator conditions of multiplicative (generalized) reverse derivations [20]. Recently, Golbasi et.al.[7] proved some results of generalized (α, β) derivations on Jordan ideals in ∗-prime rings. In the present paper, we shall study these results on Jordan ideals and generalized (α, 1)- reverse derivations in ∗-prime rings.

2. PRELIMINARIES

Throughout this paper, R will denote an associative ring with center Z(R). We will write for all x, y ∈ R, [x, y] = xy − yx and xoy = xy + yx for the Lie product and Jordan product, respectively. R is 2-torsion free if whenever 2x = 0, with x ∈ R, then x = 0. R is prime if aRb = 0 implies a = 0 or b = 0. If R admits an involution ∗, then R is ∗-prime if $aRb = aRb^* = 0$ or $aRb = a^*Rb = 0$ then a = 0 or b = 0. An additive subgroup J of R is said to be a Jordan ideal of R if uor ∈ J, for all u ∈ J and r ∈ R. A Jordan ideal J which satisfies $J^* = J$ is called a ∗-Jordan ideal. A reverse derivation d commutes with an involution ∗ if $d(r^*) = (d(r))^*$, for all r ∈ R. An additive mapping d: R → R is called a reverse derivation if d(xy) = d(y)x + yd(x) holds for all x, y ∈ R. An additive mapping F: R → R is called a generalized reverse derivation if there exists a derivation d: R → R such that F(xy) = F(y)x + yd(x), for all x, y ∈ R. Suppose that α be an auto-morphism of R. An additive mapping d: R→ R is said to be a (α, 1)- reverse derivation of R if d(xy) = d(y)α(x) + yd(x), for all x, y ∈ R. An additive mapping F: R → R is called a generalized (α, 1)- reverse derivation if there exists a reverse derivation d: R → R such that F(xy) = F(y)α(x) + yd(x), for all x, y ∈ R. Let S be a nonempty subset of R. A mapping F: R→ R is called centralizing on S if [F(x), x] ∈ Z, for all x ∈ S and is called commuting on S if [F(x), x] = 0, for all x ∈ S. Throughout, (R, ∗) will be a 2- torsion free *-prime ring, α be automorphism of R and $Sa_*(R) = \{r \in R / r^* = \pm r\}$ the set of symmetric and skew symmetric elements of R. F be a nonzero generalized (α, 1)- reverse derivation of R with associated nonzero (α, 1)- reverse derivation d which commutes with ∗ and J be a nonzero *-Jordan ideal and a subring of R. Also, we will make some extensive use of the basic commutator identities:

$$[x, yz] = y[x, z] + [x, y]z; \ [xy, z] = [x, z]y + x[y, z];$$

$$[xy, z]_{\alpha,1} = x[y, z]_{\alpha,1} + [x, z]y = x[y, \alpha(z)] + [x, z]_{\alpha,1}y;$$

$$[x, yz]_{\alpha,1} = y[x, z]_{\alpha,1} + [x, y]_{\alpha,1}\alpha(z);$$

$$xo(yz) = (xoy)z - y[x, z]= y(xoz) + [x, y]z;$$

$$(xy)oz = x(yoz) - [x, z]y = (xoz)y + x [y, z];$$

$$(xo(yz))_{\alpha,1} = (xoy)_{\alpha,1}\alpha(z) - y[x, z]_{\alpha,1} = y(xoz)_{\alpha,1} + [x, y]_{\alpha,1}\alpha(z);$$

$$((xy)oz)_{\alpha,1} = x(yoz)_{\alpha,1} - [x, z]y = (xoz)_{\alpha,1}y + x [y, \alpha(z)].$$

Lemma 2.1 ([16, Lemma 2]): Let R be a 2-torsion free ∗-prime ring and J a nonzero ∗-Jordan ideal of R. If aJb = a*Jb = 0, then a = 0 or b = 0.

Lemma 2.2 ([16, Lemma 3]): Let R be a 2-torsion free ∗-prime ring and J a nonzero ∗-Jordan ideal of R. If [J, J] = 0, then J ⊆ Z(R).

Lemma 2.3 ([17, Lemma 3]): Let R be a 2-torsion free ∗-prime ring and J a nonzero ∗-Jordan ideal of R. If J ⊆ Z(R), then R is commutative.

Lemma 2.4: Let R be a 2-torsion free ∗-prime ring, J a nonzero ∗-Jordan ideal of R and d a non- zero (α, 1)- reverse derivation of R. If d commutes with ∗ and d(J) = 0, then R is commutative.

Proof: By the hypothesis, we have d(uv) = d(v)α(u) + vd(u), for all u, v ∈ J.

From d(uor) = 0, for all u ∈ J, r ∈ R.

To expanding and using the hypothesis, we get $(d(r)ou)_{\alpha,1} = 0$, for all u ∈ J, r ∈ R. (2.1)

Replacing r by 2vr, v ∈ J in (2.1) and using (2.1), d(J) = 0, we have

d(r)α([v, u]) = 0, for all u, v ∈ J, r ∈ R. (2.2)

Replacing r by sr, s ∈ R in (2.2) and using (2.2), we find that d(r)α(s)α([v, u]) = 0.

Thus d(r)Rα([v, u]) = 0, for all u, v ∈ J, r ∈ R. (2.3)

Replacing r by r* in (2.3), we get d(r∗)Rα([v, u]) = 0, for all u, v ∈ J, r ∈ R.

Since d commutes with ∗ and J* = J, the last equation yields

d(r)*Rα([v, u]) = 0, for all u, v ∈ J, r ∈ R. (2.4)

By the ∗-primeness and α be automorphism of R, from (2.3) and (2.4), we conclude that d(r) = 0 or [v, u] = 0, for all u, v ∈ J, r ∈ R.

Since d is a nonzero (α, 1)- reverse derivation of R, we arrive that [J, J] = 0, and so R is commutative by Lemmas 2.2 and 2.3.

3. MAIN RESULTS

Theorem 3.1: Let R be a 2-torsion free *-prime ring and J a nonzero *-Jordan ideal and a subring of R. If R admits a nonzero generalized (α, 1)- reverse derivation F associated with nonzero (α, 1)- reverse derivation d which commutes with * such that $[F(u), u]_{\alpha,1} = 0$, for all u∈ J, then R is commutative.

Proof: Suppose that $[F(u), u]_{\alpha,1} = 0$, for all $u \in J$. (3.1)

Linearizing (3.1) and using this, we obtain that

$[F(u), v]_{\alpha,1} + [F(v), u]_{\alpha,1} = 0$, for all $u, v \in J$. (3.2)

Replacing v by uv in (3.2), we get $[F(u), uv]_{\alpha,1} + [F(v)\alpha(u) + vd(u), u]_{\alpha,1} = 0$.

Expanding this by using (3.1) and (3.2) in the above relation, we get

$v[d(u), u]_{\alpha,1} + [v, u]d(u) = 0$. (3.3)

Replacing v by vw in (3.3) and using (3.3), we get $[v, u]wd(u) = 0$ and thus

$[v, u]Jd(u) = 0$, for all $u, v, w \in J$. (3.4)

Since J is a nonzero $*$-Jordan ideal of R yields that

$[v, u]^*J(d(u)) = 0$, for all $v \in J$, $u \in J \cap Sa_*(R)$. (3.5)

From (3.4) and (3.5), we get $[v, u]Jd(u) = [v, u]^*Jd(u) = 0$. (3.6)

By Lemma 2.1, we get either $[v, u] = 0$, for all $v \in J$ or $d(u) = 0$, for each $u \in J \cap Sa_*(R)$.

Let $u \in J$, as $u \pm u^* \in J \cap Sa_*(R)$ and $[v, u \pm u^*] = 0$, for all $v \in J$ or $d(u \pm u^*) = 0$.

Hence we have $[v, u] = 0$ or $d(u) = 0$, for all $u, v \in J$.

Consider $J_1 = \{u \in J/\ d(u) = 0\}$ and $J_2 = \{v \in J/\ [v, u] = 0\}$; it is clear that J_1 and J_2 are additive subgroups of J such that $J = J_1 \cup J_2$. But a group cannot be a union of two of its proper subgroups so that $J = J_1$ or $J = J_2$.

If $J = J_1$, then $d(J) = 0$ and lemma 2.4 forces either $d = 0$ or $J \subseteq Z(R)$. In the former case, we get R is commutative by Lemma 2.4.

If $J = J_2$, then $[v, u] = 0$, for all $u, v \in J$. That is $[J, J] = (0)$. Which means $J \subseteq Z$ by Lemma 2.2 and R is commutative by Lemma 2.3.

Hence the proof is completed.

Corollary 3.2: Let R be a 2-torsion free *-prime ring and J a nonzero *-Jordan ideal and a subring of R. If R admits a nonzero (α, 1)- reverse derivation d which commutes with * such that $[d(u), u]_{\alpha,1} = 0$, for all $u \in J$, then R is commutative.

Theorem 3.3: Let R be a 2-torsion free *-prime ring and J a nonzero *-Jordan ideal and a subring of R. If R admits a nonzero generalized (α, 1)- reverse derivation F associated with nonzero (α, 1)- reverse derivation d which commutes with * such that $F(u)\alpha(u) = ud(u)$, for all $u \in J$, then R is commutative.

Proof: We have $F(u)\alpha(u) = ud(u)$, for all $u \in J$. (3.7)

Replacing u by u + v in (3.7) and using (3.7), we get

$F(u)\alpha(v) + F(v)\alpha(u) = ud(v) + vd(u)$. (3.8)

Replacing u by vu in (3.8) and using (3.8), we obtain that

$2ud(v)\alpha(v) = (vou)d(v)$, for all u, v∈ J. (3.9)

Replacing u by wu in (3.9) and using (3.9), we obtain that $[w, v]ud(v) = 0$, for all u, v, w ∈ J.

Thus $[w, v]Jd(v) = 0$, for all v, w ∈ J. (3.10)

Since J is a nonzero ∗-Jordan ideal of R yields that

$[w, v]^*Jd(v) = 0$, for all w ∈ J, v ∈ J∩$Sa_*(R)$.

Therefore, we get $[w, v]Jd(v) = [w, v]^*Jd(v) = 0$, for all w ∈ J, v ∈ J∩$Sa_*(R)$.

The similar arguments as used after equation (3.6), we get the required result.

Theorem 3.4: Let R be a 2-torsion free *-prime ring and J a nonzero *-Jordan ideal and a subring of R. If R admits a nonzero generalized (α, 1)- reverse derivation F associated with nonzero (α, 1)- reverse derivation d which commutes with * such that $F(u^2) = \pm\, \alpha(u^2)$, for all u∈ J, then R is commutative.

Proof: From the hypothesis $F(uu) = \alpha(u + u)$, for u∈ J. Linearizing this we get

$F(v)\alpha(u) + vd(u) + F(u)\alpha(v) + ud(v) = \alpha(uv + vu)$, for all u, v ∈ J. (3.11)

Replacing u by vu, v ∈ J in (3.11) and using (3.11), we arrive at

$(vou)d(v) = 0$, for all u, v ∈ J. (3.12)

Replacing u by uw in (3.12) and using (3.12), we obtain that $[w, v]ud(v) = 0$ and thus $[w, v]Jd(v) = 0$, for all u, v, w ∈ J.

The similar arguments as used after equation (3.10), we get the required result.

Theorem 3.5: Let R be a 2-torsion free *-prime ring and J a nonzero *-Jordan ideal and a subring of R. If R admits a nonzero generalized (α, 1)- reverse derivation F associated with nonzero(α, 1)- reverse derivation d which commutes with * such that $F(u^2) = 2d(u)\alpha(u)$, for all u∈ J, then R is commutative.

Proof: We have $F(u^2) = 2d(u)\alpha(u)$, for all u ∈ J.

i.e., $F(u)\alpha(u) + ud(u) = 2d(u)\alpha(u)$, for all u ∈ J. (3.13)

Linearizing (3.13) and using this, we obtain

$F(u)\alpha(v) + F(v)\alpha(u) + ud(v) + vd(u) = 2d(u)\alpha(v) + 2d(v)\alpha(u)$, for all u, v ∈ J. (3.14)

Replacing u by vu in (3.14) and using (3.14), we have $(uov)d(u) = 2ud(v)\alpha(v)$. (3.15)

Replacing u by wu in (3.15) and using (3.15), we get $[v, w]Jd(v) = 0$, for all v, w ∈ J.

Since J is a nonzero ∗-Jordan ideal of R yields that

$[v, w]^*Jd(v) = 0$, for all w ∈ J, v ∈ J∩$Sa_*(R)$.

Therefore, we get $[v, w]Jd(v) = [v, w]^*Jd(v) = 0$, for all w ∈ J, v ∈ J∩$Sa_*(R)$.

The similar arguments as used after equation (3.6), we get the required result.

Theorem 3.6: Let R be a 2-torsion free *-prime ring and J a nonzero *-Jordan ideal and a subring of R. If R admits a nonzero generalized (α, 1)- reverse derivation F associated with nonzero (α, 1)- reverse derivation d which commutes with * such that $d(u^2) = 2F(u)\alpha(u)$, for all u∈ J, then R is commutative.

Proof: We have $d(u^2) = 2F(u)\alpha(u)$, for all u ∈ J.

That is $d(u)\alpha(u) + ud(u) = 2F(u)\alpha(u)$, for all u ∈ J. (3.16)

Linearizing (3.16) and using this, we obtain

$d(u)\alpha(v) + d(v)\alpha(u) + ud(v) + vd(u) = 2F(u)\alpha(v) + 2F(v)\alpha(u)$, for all u, v ∈ J. (3.17)

Replacing u by vu in (3.17) and using (3.17), we have $(v\circ u)d(v) = 2ud(v)\alpha(v)$. (3.18)

The similar arguments as used after equation (3.15) in the proof of theorem 3.5, we get the required result.

We can give the following corollary by theorem 3.5 or 3.6.

Corollary 3.7: Let R be a 2-torsion free *-prime ring and J a nonzero *-Jordan ideal and a subring of R. If R admits a nonzero (α, 1)- reverse derivation d which commutes with * such that $d(u^2) = 2d(u)\alpha(u)$, for all u∈ J, then R is commutative.

REFERENCES

[1] Ashraf.M and Rehman.N: On Jordan ideals and Jordan derivations of prime rings, Demonstratio Mathematica, 37(3)(2004), 517- 523.

[2] Bell.H.E and Martindale: Centralizing mappings of semiprime rings, Can.Math.Bull., 30(1987), 92–101.

[3] Bell.H.E and Kappe.L.C: Rings in which derivations satisfy certain algebraic conditions, Acta. Math. Hungar, 53(1989), 339–346.

[4] Bresar and Vukman: On some additive mapping in rings with involution, Aeq.Math, 38(1989), 178- 185.

[5] Bresar.M: On the distance of the compositions of two derivations to the generalized derivations, Glasg. Math.J., 33(1)(1991), 89–93.

[6] Bresar.M: Centralizing mappings and derivations in prime rings, J. Algebra, 156 (1993), 385–394.

[7] Golbasi.O and Kizilgoz.O: Generalized (α, β)-derivations on Jordan ideals in *-Prime Rings, Rend. Circ. Mat. Palermo, 63(2014), 11-17.

[8] Jaya Subba Reddy.C, Mallikarjuna Rao.S and Vasantha Kumar.S: Generalized Reverse Derivations on Prime Rings, Int.J.Math.Comp.Appli.& Res., 5(3)(2015), 101-106.

[9] Jaya Subba Reddy.C, MallikarjunaRao.S and Vijaya kumar.V: Centralizing and Commuting Left Generalized Derivations on Prime Rings, Int.J.Bull.Math.Sci.& Appli., 11(2015), 1-3.

[10] Jaya Subba Reddy.C and Haseena.Sk: Homomorphism or Anti-Homomorphism of Left (α, 1) derivations in Prime rings, Int.J.Mathematical Archive, 12(5)(2021), 45-49.

[11] Jaya Subba Reddy.C and Haseena.Sk: (α, 1)-Reverse derivations on Prime near rings, Int. J. Algebra, 15(4)(2021), 165- 170.

[12] Ibraheem.A.M: Right ideal and Generalized reverse derivations on prime rings, Amer. J. Comp. Appl.& Math., 6(4)(2016), 162-164.

[13] Mayne.J.H: Centralizing automorphisms of prime rings, Can.Math.Bull., 19(1976), 113–115.

[14] Merva Ozdeir and Neset Aydin: (α, β)-Reverse derivations on Prime and Semiprime rings, Int. J. Open problems in Comp.& Math., 11(3)(2018), 48- 59.

[15] Oukhtite.L and Salhi.S: On generalized derivations of σ-prime rings, Afr.Diaspora.J. Math., 5(1)(2007), 21–25.

[16] Oukhtite.L: On Jordan ideals derivations in rings with involution, Comment.Math.Univ. Carolin, 51(3)(2010), 389- 395.

[17] Oukhtite.L: Posner's second theorem for Jordan ideals in rings with involution, Expo. Math., 29(4)(2011), 415–419.

[18] Posner.E.C, Derivations in prime rings, Proc.Amer.Math.Soc., 8(1957), 1093-1100.

[19] Samman.M and Alyamani.N: Derivations and reverse derivations in semiprime rings, Int. J. Forum, 39(2)(2007), 1895-1902.

[20] Sandhu.G.S and Kumar.D: Annihilator conditions of multiplicative (Generalized) reverse derivation on prime rings, Int.Elec.J.Algebra, 25(2019), 87-103.