Estimation of AR (2) Model with Dependent Errors for Unbounded Stationary and Nonstationary Time Series

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Abstract

In this paper the GLS and the ML estimators, the variance-covariance matrix, the unbiased for the GLS and the ML estimators of parameters of AR (2) model with constant in case of dependent errors have been derived, the simulation results shown that the values of MSE and Thiel's U in case of unbounded stationary time series for all sample size T are less than the values of MSE and Thiel's U in case of unbounded nonstationary time series which approved that the results for unbounded stationary times series are better than the results for unbounded nonstationary times series, and the simulation results for unbounded nonstationary time series shown that by using the measurement of MSE the best case among of all cases of nonstationary which gives the smallest values of MSE is case four when the first and the second conditions of stationary conditions for AR (2) model are exists, while by using the measurement of Thiel's U the best case among of all cases of nonstationary which gives the smallest values of Thiel's U is case six when the second and the third conditions of stationary conditions for AR (2) model are exists.

Keywords: AR (2) Model; GLS Estimators; ML Estimators; Mean Squared Error; Thiel's Inequality Coefficient; Unbounded Stationary Time Series; Unbounded Nonstationary Time Series.

1. Introduction

The classical regression model seeks to determine the relationship between the dependent variable and the independent variables. This regression model could be simple or multiple. However, in the linear regression model, certain assumptions are made on how a dataset will be produced by an underlying data-generating process. According to Greene (2002), these assumptions include linearity, homoscedasticity, normality, and no autocorrelation between the error terms. Moreover, the regression model describes the value of the dependent variable as the sum of two parts, a deterministic part, and a random part.

The error term is primarily a disturbance to an already stable relationship and can capture the remaining information in the dependent variable which could not be explained by the independent variables. Relating to the assumption on the error term, if the assumption of no correlation in the error term is violated, then, the underlying model would be rendered invalid with the standard errors of the parameters becoming biased. Moreover, if the errors are correlated, the least-squares estimators are inefficient and the estimated variances are not appropriate Granger and Newbold (1974) and Akpan, et al (2016).

By definition, autocorrelation is the lag correlation of a given series with itself, lagged by some time units Gujarati (2004). Thus, when applying regression models to economic/management data in the presence of autocorrelation, the ordinary least squares estimation method ceases to provide efficient estimators and appropriate variances.

The analysis of time series is very important and it is a rapidly evolving field, generally, time series is a sequence of values a specific variable has taken on over some time. The observations have a natural ordering in time. Usually, if a series of observations are referred to as a time series then some regularity of the observation frequency was assumed. Of course, the observation frequency could be more frequent than yearly. For instance, observations may be available for each quarter, each month, or even each day of a particular period. Nowadays, time series of stock prices or other financial market variables are even available at a much higher frequency such as every few minutes or seconds, Lütkepohl and Krätzig (2004).

A model often specified for the generation of an economic time series is the stochastic difference equation with independently and identically distributed errors. In practice, the most common assumption is that the time series is stationary. However, there are situations in which the stationarity assumption is not appropriate. Two such situations are testing the random walk hypothesis or unit root hypothesis and testing the first difference hypothesis, Evans and Savin (1981).

For the second-order autoregressive AR (2) model in the case of real roots, the stationarity conditions for an AR (2) processes are as follows:

$$\left.\begin{array}{l} 1)\ \rho_1 + \rho_2 < 1 \\ 2)\ \rho_2 - \rho_1 < 1 \\ 3)\ |\rho_2| < 1 \end{array}\right\} \quad (1)$$

where ρ_1 and ρ_2 are the autoregressive coefficients of the AR (2) model, David (2012)

Autoregressive time series with a unit root has been the subject of much recent attention in the econometrics literature. In part, this is because the unit root hypothesis is of considerable interest in applications, not only with data from financial and commodity markets where it has a long history but also with aggregate time series, Phillips (1987).

In an attempt to overcome the weaknesses of the ordinary least squares estimation method in the presence of autocorrelation, this study seeks to apply the generalized least squares estimation method since the least-squares estimation method does not make use of the information of the unexplained variance as captured by the error terms in the dependent variable, whereas the generalized least squares (GLS) takes such information, the unexplained variance into account explicitly and is accomplished, Akpan and Moffat (2018).

In this paper, the GLS and the ML estimators, the variance-covariance matrix for the GLS and ML estimators of parameters of AR (2) model with constant in case of dependent errors will be derived and a simulation study in two cases unbounded stationary time series and unbounded nonstationary time series will be derived.

2. Model and Assumptions

The second-order autoregressive AR (2) with a constant model in case of dependent errors takes the following form:

$$y_t = \alpha + \rho_1 y_{t-1} + \rho_2 y_{t-2} + u_t, \qquad t = 1, \ldots, T, \quad (2)$$

where y_t is time series, T is the sample size, $y_0 = y_{-1} = 0$, u_t are dependent error terms ρ_1 and ρ_2 are the autoregressive coefficients, and α is the constant term.

Model (2) can be represented in matrix form as follows:

$$Y = X\,\beta + \mathbf{u} \tag{3}$$

Where:

$$\beta = \begin{bmatrix} \alpha \\ \rho_1 \\ \rho_2 \end{bmatrix},\; X = \begin{bmatrix} 1 & y_0 & y_{-1} \\ 1 & y_1 & y_0 \\ 1 & y_2 & y_1 \\ \vdots & \vdots & \vdots \\ 1 & y_{T-1} & y_{T-2} \end{bmatrix},\; Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_T \end{bmatrix} \text{ and } \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_T \end{bmatrix} \tag{4}$$

By assuming that u_t are dependent error terms which generating by the first-order stationary autoregressive process AR (1) model then, it will be as follows:

$$u_t = \phi_1 u_{t-1} + e_t \,,\, |\phi_1| < 1 \,,\, t = 1, \ldots, T \tag{5}$$

Where the error terms e_t are i.i.d. $N(0, \sigma_e^2)$ and achieved the following assumptions:

$$\left.\begin{array}{ll} 1)\, E(e_t) = 0 & \forall t \\ 2)\, \sigma_e^2 = E(e_t^2) = \sigma^2 & \forall t \\ 3)\, \sigma_{s\,t} = Cov(e_t, e_s) = E(e_t\, e_s) = 0 & \forall t \neq s \end{array}\right\} \tag{6}$$

To the sample interval $t = 1, \ldots, T$ the variance-covariance matrix for the vector of error terms $\mathbf{u} = [u_1, u_2, \ldots, u_T]$ for model (2) can be obtained by using the lag operator (L) as follows:

From equation (5) $u_t = \phi_1 u_{t-1} + e_t$ then:

$$u_t - \phi_1 u_{t-1} = e_t$$

$$(1 - \phi_1 L)\, u_t = e_t$$

$$u_t = (1 - \phi_1 L)^{-1} e_t = \sum_{j=0}^{\infty} \phi_1^{\,j} L^j e_t = \sum_{j=0}^{\infty} \phi_1^{\,j} e_{t-j} \,,\, |\phi_1| < 1 \tag{7}$$

Then $E(u_t), Var(u_t)$ and $Cov(u_t, u_{t-s})$ will be as follows:

i) $E(u_t) = 0$ (8)

ii) $Var(u_t) = \frac{\sigma^2}{(1-\phi_1^2)}$ (9)

iii) $Cov(u_t, u_{t-s}) = \sigma^2 \phi_1^s (\frac{1}{1-\phi_1^2}), s = 1, 2, 3, ...$ (10)

Then, by using equations (9) and (10), the variance-covariance matrix for the vector of error terms $\mathbf{u}=[u_1, u_2, ..., u_T]$ will be:

$$Cov(\mathbf{u}) = E\left\{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_T \end{bmatrix}\begin{bmatrix} u_1 & u_2 & u_3 \cdots u_T \end{bmatrix}\right\} = \sigma^2 \Omega \quad (11)$$

Where:

$$\Omega = \frac{1}{(1-\phi_1^2)}\begin{bmatrix} 1 & \phi_1 & \phi_1^2 & \cdots & \phi_1^{T-1} \\ \phi_1 & 1 & \phi_1 & \cdots & \phi_1^{T-2} \\ \phi_1^2 & \phi_1 & 1 & \cdots & \phi_1^{T-3} \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{T-1} & \phi_1^{T-2} & \phi_1^{T-3} & \cdots & 1 \end{bmatrix} \quad (12)$$

Fox and Weisberg (2018)

3. GLS Estimation for AR (2) Model in Case of Dependent Errors

In this section, the GLS estimators, unbiased of GLS estimators and the variance-covariance matrix for GLS estimators for the parameters of AR (2) model with constant in case of dependent errors will be derived.

3.1 The GLS Estimators for the Parameters of AR (2) Model

The GLS estimators for the parameters of AR (2) with a constant model in case of dependent errors as in equation (3) will be derived according to the following lemma:

Lemma (1): The GLS estimators for the parameters of AR (2) with a constant model in case of dependent errors as in equation (3) under the assumptions of the model will be as follows:

$$\tilde{\alpha}=\frac{K_{11}N_{11}+K_{12}N_{12}+K_{13}N_{13}}{G}$$

$$\tilde{\rho}_1=\frac{K_{21}N_{11}+K_{22}N_{12}+K_{23}N_{13}}{G}$$

$$\tilde{\rho}_2=\frac{K_{31}N_{11}+K_{32}N_{12}+K_{33}N_{13}}{G}$$

Where:

$K_{11}=CF-E^2$, $K_{22}=\Delta F-B^2$, $K_{33}=\Delta C-A^2$, $K_{12}=K_{21}=BE-AF$

$K_{32}=K_{23}=AB-\Delta E$ and $K_{13}=K_{31}=AE-CB$

$$G=\Delta CF-\Delta E^2-A^2F+2ABE-B^2C$$

$$\Delta=1-\phi_1+1-2\phi_1+\phi_1^2+1-2\phi_1+\phi_1^2+\cdots+1-\phi_1=T-2(T-1)\phi_1+(T-2)\phi_1^2$$

$$A=(1-\phi_1)y_0+(1-2\phi_1+\phi_1^2)y_1+(1-2\phi_1+\phi_1^2)y_2+\cdots+(1-\phi_1)y_{T-1}$$

$$=(1-\phi_1)y_0+(1-\phi_1)y_{T-1}+(1-2\phi_1+\phi_1^2)\sum_{t=1}^{T-2}y_t$$

$$=[y_{-1}-\phi_1y_0]y_1+[y_{T-3}-\phi_1y_{T-2}]y_T+\sum_{t=1}^{T-2}[-\phi_1(y_{t-2}+y_t)+(1+\phi_1^2)y_{t-1}]y_{t+1}$$

$$B=(1-\phi_1)y_{-1}+(1-2\phi_1+\phi_1^2)y_0+(1-2\phi_1+\phi_1^2)y_1+\cdots+(1-\phi_1)y_{T-2}$$

$$=(1-\phi_1)y_{-1}+(1-\phi_1)y_{T-2}+(1-2\phi_1+\phi_1^2)\sum_{t=1}^{T-2}y_{t-1}$$

$$C=(y_0-\phi_1y_1)y_0+[-\phi_1(y_0+y_2)+(1+\phi_1^2)y_1]y_1+[-\phi_1(y_1+y_3)+(1+\phi_1^2)y_2]y_2$$
$$+\cdots+[y_{T-2}-\phi_1y_{T-1}]y_{T-1}$$

$$=(y_0-\phi_1y_1)y_0+[y_{T-2}-\phi_1y_{T-1}]y_{T-1}+\sum_{t=1}^{T-2}[-\phi_1(y_{t-1}+y_{t+1})+(1+\phi_1^2)y_t]y_t$$

$$E=(y_0-\phi_1y_1)y_{-1}+[-\phi_1(y_0+y_2)+(1+\phi_1^2)y_1]y_0+[-\phi_1(y_1+y_3)+(1+\phi_1^2)y_2]y_1$$
$$+\cdots+[y_{T-2}-\phi_1y_{T-1}]y_{T-2}$$

$$=(y_0-\phi_1y_1)y_{-1}+[y_{T-2}-\phi_1y_{T-1}]y_{T-2}+\sum_{t=1}^{T-2}[-\phi_1(y_{t-1}+y_{t+1})+(1+\phi_1^2)y_t]y_{t-1}$$

$$F=(y_{-1}-\phi_1y_0)y_{-1}+[-\phi_1(y_{-1}+y_1)+(1+\phi_1^2)y_0]y_0+[-\phi_1(y_0+y_2)+(1+\phi_1^2)y_1]y_1$$
$$+\cdots+[y_{T-3}-\phi_1y_{T-2}]y_{T-2}$$

$$=(y_{-1}-\phi_1y_0)y_{-1}+[y_{T-3}-\phi_1y_{T-2}]y_{T-2}+\sum_{t=1}^{T-2}[-\phi_1(y_{t-2}+y_t)+(1+\phi_1^2)y_{t-1}]y_{t-1}$$

$$N_{11}=(1-\phi_1)y_1+(1-2\phi_1+\phi_1^2)y_2+(1-2\phi_1+\phi_1^2)y_3+\cdots+(1-\phi_1)y_T$$

$$=(1-\phi_1)y_1+(1-\phi_1)y_T+(1-2\phi_1+\phi_1^2)\sum_{t=1}^{T-2}y_{t+1}$$

$$N_{12}=[y_0-\phi_1y_1]y_1+[-\phi_1(y_0+y_2)+(1+\phi_1^2)y_1]y_2+[-\phi_1(y_1+y_3)+(1+\phi_1^2)y_2]y_3$$
$$+\cdots+[y_{T-2}-\phi_1y_{T-1}]y_T$$

$$=[y_0-\phi_1y_1]y_1+[y_{T-2}-\phi_1y_{T-1}]y_T+\sum_{t=1}^{T-2}[-\phi_1(y_{t-1}+y_{t+1})+(1+\phi_1^2)y_t]y_{t+1}$$

$$N_{13} = [y_{-1} - \phi_1 y_0] y_1 + [-\phi_1(y_{-1} + y_1) + (1+\phi_1^2) y_0] y_2 + [-\phi_1(y_0 + y_2) + (1+\phi_1^2) y_1] y_3 + \cdots + [y_{T-3} - \phi_1 y_{T-2}] y_T$$
$$= [y_{-1} - \phi_1 y_0] y_1 + [y_{T-3} - \phi_1 y_{T-2}] y_T + \sum_{t=1}^{T-2} [-\phi_1(y_{t-2} + y_t) + (1+\phi_1^2) y_{t-1}] y_{t+1}$$

Proof:

The GLS Estimator $\tilde{\beta}$, in general, is defined as follows:

$$\tilde{\beta} = (X'\Omega^{-1}X)^{-1} X'\Omega^{-1}Y \tag{13}$$

Fox and Weisberg (2018)

Where for model in equation (3),

$$\tilde{\beta} = \begin{bmatrix} \tilde{\alpha} \\ \tilde{\rho}_1 \\ \tilde{\rho}_2 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & y_0 & y_{-1} \\ 1 & y_1 & y_0 \\ \vdots & \vdots & \vdots \\ 1 & y_{T-1} & y_{T-2} \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix} \tag{14}$$

By substituting from equations (12) and (14) in equation (13) then the GLS estimator $\tilde{\beta}$ for the model in equation (3) will be as follows:

$$\begin{bmatrix} \tilde{\alpha} \\ \tilde{\rho}_1 \\ \tilde{\rho}_2 \end{bmatrix} = \left\{ \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ y_0 & y_1 & y_2 & \cdots & y_{T-1} \\ y_{-1} & y_0 & y_1 & \cdots & y_{T-2} \end{bmatrix} \left(\frac{1}{(1-\phi_1^2)} \begin{bmatrix} 1 & \phi_1 & \phi_1^2 & \cdots & \phi_1^{T-1} \\ \phi_1 & 1 & \phi_1 & \cdots & \phi_1^{T-2} \\ \phi_1^2 & \phi_1 & 1 & \cdots & \phi_1^{T-3} \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{T-1} & \phi_1^{T-2} & \phi_1^{T-3} & \cdots & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & y_0 & y_{-1} \\ 1 & y_1 & y_0 \\ 1 & y_2 & y_1 \\ \vdots & \vdots & \vdots \\ 1 & y_{T-1} & y_{T-2} \end{bmatrix} \right\}^{-1}$$
$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ y_0 & y_1 & y_2 & \cdots & y_{T-1} \\ y_{-1} & y_0 & y_1 & \cdots & y_{T-2} \end{bmatrix} \left(\frac{1}{(1-\phi_1^2)} \begin{bmatrix} 1 & \phi_1 & \phi_1^2 & \cdots & \phi_1^{T-1} \\ \phi_1 & 1 & \phi_1 & \cdots & \phi_1^{T-2} \\ \phi_1^2 & \phi_1 & 1 & \cdots & \phi_1^{T-3} \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{T-1} & \phi_1^{T-2} & \phi_1^{T-3} & \cdots & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_T \end{bmatrix}$$

Then,

$$\begin{bmatrix} \tilde{\alpha} \\ \tilde{\rho}_1 \\ \tilde{\rho}_2 \end{bmatrix} = \begin{Bmatrix} \Delta & A & B \\ A & C & E \\ B & E & F \end{Bmatrix}^{-1} \begin{bmatrix} N_{11} \\ N_{12} \\ N_{13} \end{bmatrix}$$

$$\left. \begin{aligned} \tilde{\alpha} &= \frac{K_{11} N_{11} + K_{12} N_{12} + K_{13} N_{13}}{G} \\ \tilde{\rho}_1 &= \frac{K_{21} N_{11} + K_{22} N_{12} + K_{23} N_{13}}{G} \\ \tilde{\rho}_2 &= \frac{K_{31} N_{11} + K_{32} N_{12} + K_{33} N_{13}}{G} \end{aligned} \right\} \quad (15)$$

Where:

$\Delta, A, B, C, E, F, G, N_{11}, N_{12}, N_{13}, K_{11}, K_{22}, K_{33}, K_{12} = K_{21}, K_{32} = K_{23}$ and $K_{13} = K_{31}$ are defined as in lemma (1).

3.2 The Unbiased for GLS Estimators of Parameters of AR (2) Model

From equation (13) the GLS estimators for parameters of AR (2) model with constant in case of dependent errors is as follows:

$$\tilde{\beta} = (X'\Omega^{-1}X)^{-1} X'\Omega^{-1}Y$$

Then,

$$\begin{aligned} E(\tilde{\beta}) &= (X'\Omega^{-1}X)^{-1} X'\Omega^{-1}E(Y) \\ &= (X'\Omega^{-1}X)^{-1} X'\Omega^{-1}E(X'\beta + \mathbf{u}) \\ &= (X'\Omega^{-1}X)^{-1} X'\Omega^{-1}X'\beta + E(\mathbf{u}) = \beta + E(\mathbf{u}) \end{aligned}$$

By using equation (8) then,

$$E(\tilde{\beta}) = \beta + E(\mathbf{u}) = \beta + \mathbf{0} = \beta$$

Therefore the GLS estimators $\tilde{\beta}$ are unbiased estimators for β.

3.3 Variance-Covariance Matrix for GLS Estimators for AR (2) Model

In this section the variance-covariance matrix for GLS estimators $\tilde{\beta}$ of parameters of AR (2) model with constant in case of dependent errors will be obtained according to the following lemma:

Lemma (2): The variance-covariance matrix for GLS estimators $\tilde{\beta}$ of parameters of AR (2) model with constant in case of dependent errors as in equation (3) under the assumptions of the model will be as follows:

$$\begin{bmatrix} V(\tilde{\alpha}) & Cov(\tilde{\alpha},\tilde{\rho}_1) & Cov(\tilde{\alpha},\tilde{\rho}_2) \\ Cov(\tilde{\alpha},\tilde{\rho}_1) & V(\tilde{\rho}_1) & Cov(\tilde{\rho}_1,\tilde{\rho}_2) \\ Cov(\tilde{\alpha},\tilde{\rho}_2) & Cov(\tilde{\rho}_1,\tilde{\rho}_2) & V(\tilde{\rho}_2) \end{bmatrix} = \frac{\sigma^2}{G}\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}$$

Where:

$K_{11} = CF - E^2$, $K_{22} = \Delta F - B^2$, $K_{33} = \Delta C - A^2$, $K_{12} = K_{21} = BE - AF$

$K_{32} = K_{23} = AB - \Delta E$ and $K_{13} = K_{31} = AE - CB$, $G = \Delta CF - \Delta E^2 - A^2 F + 2ABE - B^2 C$

$$\Delta = 1-\phi_1 + 1 - 2\phi_1 + \phi_1^2 + 1 - 2\phi_1 + \phi_1^2 + \cdots + 1 - \phi_1 = T - 2(T-1)\phi_1 + (T-2)\phi_1^2$$

$$A = (1-\phi_1)y_0 + (1-2\phi_1+\phi_1^2)y_1 + (1-2\phi_1+\phi_1^2)y_2 + \cdots + (1-\phi_1)y_{T-1}$$
$$= (1-\phi_1)y_0 + (1-\phi_1)y_{T-1} + (1-2\phi_1+\phi_1^2)\textstyle\sum_{t=1}^{T-2} y_t$$
$$= [y_{-1} - \phi_1 y_0]y_1 + [y_{T-3} - \phi_1 y_{T-2}]y_T + \textstyle\sum_{t=1}^{T-2}[-\phi_1(y_{t-2}+y_t) + (1+\phi_1^2)y_{t-1}]y_{t+1}$$

$$B = (1-\phi_1)y_{-1} + (1-2\phi_1+\phi_1^2)y_0 + (1-2\phi_1+\phi_1^2)y_1 + \cdots + (1-\phi_1)y_{T-2}$$
$$= (1-\phi_1)y_{-1} + (1-\phi_1)y_{T-2} + (1-2\phi_1+\phi_1^2)\textstyle\sum_{t=1}^{T-2} y_{t-1}$$

$$C = (y_0 - \phi_1 y_1)y_0 + [-\phi_1(y_0+y_2) + (1+\phi_1^2)y_1]y_1 + [-\phi_1(y_1+y_3) + (1+\phi_1^2)y_2]y_2$$
$$+\cdots+ [y_{T-2} - \phi_1 y_{T-1}]y_{T-1}$$
$$= (y_0 - \phi_1 y_1)y_0 + [y_{T-2} - \phi_1 y_{T-1}]y_{T-1} + \textstyle\sum_{t=1}^{T-2}[-\phi_1(y_{t-1}+y_{t+1}) + (1+\phi_1^2)y_t]y_t$$

$$E = (y_0 - \phi_1 y_1)y_{-1} + [-\phi_1(y_0+y_2) + (1+\phi_1^2)y_1]y_0 + [-\phi_1(y_1+y_3) + (1+\phi_1^2)y_2]y_1$$
$$+\cdots+ [y_{T-2} - \phi_1 y_{T-1}]y_{T-2}$$
$$= (y_0 - \phi_1 y_1)y_{-1} + [y_{T-2} - \phi_1 y_{T-1}]y_{T-2} + \textstyle\sum_{t=1}^{T-2}[-\phi_1(y_{t-1}+y_{t+1}) + (1+\phi_1^2)y_t]y_{t-1}$$

$$F = (y_{-1} - \phi_1 y_0)y_{-1} + [-\phi_1(y_{-1}+y_1) + (1+\phi_1^2)y_0]y_0 + [-\phi_1(y_0+y_2) + (1+\phi_1^2)y_1]y_1$$
$$+\cdots+ [y_{T-3} - \phi_1 y_{T-2}]y_{T-2}$$
$$= (y_{-1} - \phi_1 y_0)y_{-1} + [y_{T-3} - \phi_1 y_{T-2}]y_{T-2} + \textstyle\sum_{t=1}^{T-2}[-\phi_1(y_{t-2}+y_t) + (1+\phi_1^2)y_{t-1}]y_{t-1}$$

Proof:

The variance-covariance matrix for GLS estimators in general is as follows:

$$V(\tilde{\beta}) = \sigma^2 (X'\Omega^{-1}X)^{-1} \qquad (16)$$

Fox and Weisberg (2018)

Where for model in equation (3),

$$V(\tilde{\beta})=\begin{bmatrix} V(\tilde{\alpha}) & Cov(\tilde{\alpha},\tilde{\rho}_1) & Cov(\tilde{\alpha},\tilde{\rho}_2) \\ Cov(\tilde{\alpha},\tilde{\rho}_1) & V(\tilde{\rho}_1) & Cov(\tilde{\rho}_1,\tilde{\rho}_2) \\ Cov(\tilde{\alpha},\tilde{\rho}_2) & Cov(\tilde{\rho}_1,\tilde{\rho}_2) & V(\tilde{\rho}_2) \end{bmatrix},\ X=\begin{bmatrix} 1 & y_0 & y_{-1} \\ 1 & y_1 & y_0 \\ 1 & y_2 & y_1 \\ \vdots & \vdots & \vdots \\ 1 & y_{T-1} & y_{T-2} \end{bmatrix},$$

$$\Omega=\frac{1}{(1-\phi_1^2)}\begin{bmatrix} 1 & \phi_1 & \phi_1^2 & \cdots & \phi_1^{T-1} \\ \phi_1 & 1 & \phi_1 & \cdots & \phi_1^{T-2} \\ \phi_1^2 & \phi_1 & 1 & \cdots & \phi_1^{T-3} \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{T-1} & \phi_1^{T-2} & \phi_1^{T-3} & \cdots & 1 \end{bmatrix} \tag{17}$$

By substituting from equation (17) in equation (16) then the $V(\tilde{\beta})$ for model in equation (3) will be as follows:

$$\begin{bmatrix} V(\tilde{\alpha}) & Cov(\tilde{\alpha},\tilde{\rho}_1) & Cov(\tilde{\alpha},\tilde{\rho}_2) \\ Cov(\tilde{\alpha},\tilde{\rho}_1) & V(\tilde{\rho}_1) & Cov(\tilde{\rho}_1,\tilde{\rho}_2) \\ Cov(\tilde{\alpha},\tilde{\rho}_2) & Cov(\tilde{\rho}_1,\tilde{\rho}_2) & V(\tilde{\rho}_2) \end{bmatrix}=$$

$$\sigma^2\left\{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ y_0 & y_1 & y_2 & \cdots & y_{T-1} \\ y_{-1} & y_0 & y_1 & \cdots & y_{T-2} \end{bmatrix}\left(\frac{1}{(1-\phi_1^2)}\begin{bmatrix} 1 & \phi_1 & \phi_1^2 & \cdots & \phi_1^{T-1} \\ \phi_1 & 1 & \phi_1 & \cdots & \phi_1^{T-2} \\ \phi_1^2 & \phi_1 & 1 & \cdots & \phi_1^{T-3} \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{T-1} & \phi_1^{T-2} & \phi_1^{T-3} & \cdots & 1 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & y_0 & y_{-1} \\ 1 & y_1 & y_0 \\ 1 & y_2 & y_1 \\ 1 & \vdots & \vdots \\ 1 & y_{T-1} & y_{T-2} \end{bmatrix}\right\}^{-1}$$

Then,

$$\begin{bmatrix} V(\tilde{\alpha}) & Cov(\tilde{\alpha},\tilde{\rho}_1) & Cov(\tilde{\alpha},\tilde{\rho}_2) \\ Cov(\tilde{\alpha},\tilde{\rho}_1) & V(\tilde{\rho}_1) & Cov(\tilde{\rho}_1,\tilde{\rho}_2) \\ Cov(\tilde{\alpha},\tilde{\rho}_2) & Cov(\tilde{\rho}_1,\tilde{\rho}_2) & V(\tilde{\rho}_2) \end{bmatrix}=\sigma^2\begin{Bmatrix} \Delta & A & B \\ A & C & E \\ B & E & F \end{Bmatrix}^{-1}$$

$$\begin{bmatrix} V(\tilde{\alpha}) & Cov(\tilde{\alpha},\tilde{\rho}_1) & Cov(\tilde{\alpha},\tilde{\rho}_2) \\ Cov(\tilde{\alpha},\tilde{\rho}_1) & V(\tilde{\rho}_1) & Cov(\tilde{\rho}_1,\tilde{\rho}_2) \\ Cov(\tilde{\alpha},\tilde{\rho}_2) & Cov(\tilde{\rho}_1,\tilde{\rho}_2) & V(\tilde{\rho}_2) \end{bmatrix}=\frac{\sigma^2}{G}\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}$$

Where:

Δ, A, B, C, E, F, G, K_{11}, K_{22}, K_{33}, $K_{12}=K_{21}$, $K_{32}=K_{23}$ and $K_{13}=K_{31}$ are defined as in lemma (2).

4. The ML Estimation for AR (2) Model in Case of Dependent Errors

In this section, the ML estimators, unbiased of ML estimators and the variance-covariance matrix for ML estimators for the parameters of AR (2) model with constant in case of dependent errors will be derived.

4.1 The ML Estimators for the Parameters of AR (2) Model

The ML estimators for the parameters of AR (2) with a constant model in case of dependent errors as in equation (3) will be derived according to the following lemma:

Lemma (3): The ML estimators for the parameters of AR (2) with a constant model in case of dependent errors as in equation (3) under the assumptions of the model will be as the GLS estimators as in lemma (1).

Proof:

From equations (8) and (11) then,

$$\mathbf{u} \sim N_T(0\,,\,\sigma^2\Omega) \tag{18}$$

From equations (3) and (18) then,

$$Y \sim N_T(X\beta\,,\,\sigma^2\Omega) \tag{19}$$

By using equation (19) then,

$$\mathrm{f}\,(Y;X,\beta,\sigma^2)=(2\pi\sigma^2)^{-\frac{T}{2}}|\Omega|^{-\frac{1}{2}}\exp[-\frac{1}{2\sigma^2}(Y-X\beta)'\Omega^{-1}(Y-X\beta)] \tag{20}$$

By using equation (20) then the log of the maximum likelihood function for model in equation (3) will be as follows:

$$\ln L(\beta,\sigma^2|Y,X)=$$
$$-\frac{T}{2}\ln 2\pi-\frac{T}{2}\ln\sigma^2-\frac{1}{2}\ln|\Omega|-\frac{1}{2\sigma^2}(Y-X\beta)'\Omega^{-1}(Y-X\beta) \tag{21}$$

Fox and Weisberg (2018)

And to obtain the ML estimator $\hat{\beta}$ the following normal equation must be solved

$$\frac{\partial \ln L(\beta, \sigma^2 | Y, X)}{\partial \beta} = -\frac{1}{2\hat{\sigma}_g^2}[-X'\Omega^{-1}Y + 2(X'\Omega^{-1}X)\hat{\beta}] = \mathbf{0} \qquad (22)$$

By solving equation (22) the ML estimator $\hat{\beta}$ will be as follows:

$$\hat{\beta} = (X'\Omega^{-1}X)^{-1} X'\Omega^{-1}Y \qquad (23)$$

The form ML estimators $\hat{\beta}$ in equation (23) is the same form as GLS estimators $\tilde{\beta}$ in equation (13) and by substituting from equations (4) and (12) in equation (23) then the ML estimators $\hat{\beta}$ for AR (2) model with constant in case of dependent errors will be as GLS estimators $\tilde{\beta}$ in equation (15) and hence the ML estimators $\hat{\beta}$ are unbiased estimators for β .

4.2 Variance-Covariance Matrix for the ML Estimators for AR (2) Model

In this section the variance-covariance matrix for ML estimators $\hat{\beta}$ of parameters of AR (2) model with constant in case of dependent errors will be obtained according to the following lemma:

Lemma (4): The variance-covariance matrix for ML estimators $V(\hat{\beta})$ of parameters of AR (2) model with constant in case of dependent errors as in equation (3) under the assumptions of the model will be as the variance-covariance matrix for GLS estimators $V(\tilde{\beta})$ as in lemma (2).

Proof:

The variance-covariance matrix for ML estimators, in general, is as follows:

$$V(\hat{\beta}) = \sigma^2 (X'\Omega^{-1}X)^{-1} \qquad (24)$$

Amer (2015)

Where for model in equation (3),

$$V(\hat{\beta})=\begin{bmatrix} V(\hat{\alpha}) & Cov(\hat{\alpha},\hat{\rho}_1) & Cov(\hat{\alpha},\hat{\rho}_2) \\ Cov(\hat{\alpha},\hat{\rho}_1) & V(\hat{\rho}_1) & Cov(\hat{\rho}_1,\hat{\rho}_2) \\ Cov(\hat{\alpha},\hat{\rho}_2) & Cov(\hat{\rho}_1,\hat{\rho}_2) & V(\hat{\rho}_2) \end{bmatrix},\ X=\begin{bmatrix} 1 & y_0 & y_{-1} \\ 1 & y_1 & y_0 \\ 1 & y_2 & y_1 \\ \vdots & \vdots & \vdots \\ 1 & y_{T-1} & y_{T-2} \end{bmatrix},$$

$$\Omega=\frac{1}{(1-\phi_1^2)}\begin{bmatrix} 1 & \phi_1 & \phi_1^2 & \cdots & \phi_1^{T-1} \\ \phi_1 & 1 & \phi_1 & \cdots & \phi_1^{T-2} \\ \phi_1^2 & \phi_1 & 1 & \cdots & \phi_1^{T-3} \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{T-1} & \phi_1^{T-2} & \phi_1^{T-3} & \cdots & 1 \end{bmatrix} \tag{25}$$

By substituting from equation (25) in equation (24) then the variance-covariance matrix for ML estimators $V(\hat{\beta})$ for the model in equation (3) will be as the variance-covariance matrix for GLS estimators $V(\tilde{\beta})$ as in lemma (2).

5. Simulation Study

In this section a simulation study is used to obtain bias, MSE, Thiel's U for GLS and ML estimators of AR (2) with a constant model in case of dependent errors under two cases the first case for unbounded stationary time series and the second case for unbounded nonstationary time series as follows:

5.1. Unbounded Stationary Time Series

A simulation study is used to obtain bias, MSE, Thiel's U for GLS and ML estimators of AR (2) model with constant in case of dependent errors which obtained in equation (15) for stationary time series by using initial values for $\rho_1 = 0.95$ and $\rho_2 = 0.03$ for which the three conditions for stationarity in equation (1) exist in case of five samples size T = 30, 50, 100, 200 and 500 and ten values for the coefficient of dependent errors $\phi_1 = \pm 0.5, \pm 0.4, \pm 0.3, \pm 0.2$ and ± 0.1 by 5000 replications and the following results can be discussed for the five cases:

Case (1): T = 30

It can be noticed from table (1) in the appendix of tables that in general the values of MSE for positive values of ϕ_1 are more than the values of MSE for negative values of ϕ_1, the values of Thiel's U for positive values of ϕ_1 are less than the values of Thiel's U for negative values of ϕ_1, the values of MSE fluctuated with the descending values of ϕ_1, and the values of Thiel's U are ascending with the descending values of ϕ_1.

Case (2): T = 50

It can be noticed from the table (1) in the appendix of tables for sample size T = 50 that in general the values of MSE for positive values of ϕ_1 are more than the values of MSE for negative values of ϕ_1, the values of Thiel's U for positive values of ϕ_1 are less than the values of Thiel's U for negative values of ϕ_1, the values of MSE fluctuated with the descending values of ϕ_1, and the values of Thiel's U are ascending with the descending values of ϕ_1.

Case (3): T = 100

It can be noticed from the table (1) in the appendix of tables for sample size T = 100 that in general the values of MSE for positive values of ϕ_1 are more than the values of MSE for negative values of ϕ_1, the values of Thiel's U for positive values of ϕ_1 are less than the values of Thiel's U for negative values of ϕ_1, the values of MSE fluctuated with the descending values of ϕ_1, and the values of Thiel's U are ascending with the descending values of ϕ_1.

Case (4): T = 200

It can be noticed from the table (1) in the appendix of tables for sample size T = 200 that in general the values of MSE for positive values of ϕ_1 are more than the values of MSE for negative values of ϕ_1, the values of Thiel's U for positive values of ϕ_1 are less than the values of Thiel's U for negative values of ϕ_1, the values of MSE fluctuated with the descending values of ϕ_1, and the values of Thiel's U are ascending with the descending values of ϕ_1.

Case (5): T = 500

It can be noticed from the table (1) in the appendix of tables for sample size T = 500 that in general the values of MSE for positive values of ϕ_1 are more than the values of MSE for negative values of ϕ_1, the values of Thiel's U for positive values of ϕ_1 are less than the values of Thiel's U for negative values of ϕ_1, the values of MSE fluctuated with the descending values of ϕ_1, and the values of Thiel's U are ascending with the descending values of ϕ_1.

4.2. Unbounded Nonstationary Time Series

A simulation study is used to obtain bias, MSE, Thiel's U for GLS and ML estimators of AR (2) model with constant in case of dependent errors which obtained in equation (15) for nonstationary time series by using six pairs of initial values for ρ_1 and ρ_2 for which the three conditions for stationarity in equation (1) are non-exists in various ways in case of five sample size T = 30, 50, 100, 200 and 500 and ten values for the coefficient of dependent errors $\phi_1 = \pm 0.5, \pm 0.4, \pm 0.3, \pm 0.2$ and ± 0.1 by 5000 replications. The following results can be discussed for the next six cases:

Case 1: All Conditions of Stationary are Non-exists for Unbounded Time Series

In this case, the initial values for ρ_1 and ρ_2 are $\rho_1 = 1.2$ and $\rho_2 = 2.3$ which make all conditions of stationarity conditions for AR (2) model in equation (1) are non-exists and for unbounded time series, it can be noticed from the table (2) in the appendix of tables that for all sample sizes the values of MSE for positive values of ϕ_1 are more than the values of MSE for negative values of ϕ_1, the values of Thiel's U for positive values of ϕ_1 are less than the values of Thiel's U for negative values of ϕ_1, for sample sizes, T = 30, 50 and 100 in case of positive values of ϕ_1 the values of MSE fluctuated with the descending values of ϕ_1, for sample sizes, T = 500, 200 and 100 in case of negative values of ϕ_1 the values of MSE are descending with the descending values of ϕ_1 and for all sample sizes the values of Thiel's U are ascending with the descending values of ϕ_1.

CASE 2: The First Condition of Stationary Conditions is Exists

In this case, the initial values for ρ_1 and ρ_2 are $\rho_1 = -2$ and $\rho_2 = 1.1$ which make the first condition of stationarity conditions for AR (2) model in equation (1) exists and for unbounded time series, it can be noticed from the table (3) in the appendix of tables that for all sample sizes the values of MSE for positive values of ϕ_1 are more than the values of MSE for negative values of ϕ_1, for sample size T = 30 the values of Thiel's U for positive values of ϕ_1 are less than the values of Thiel's U for negative values of ϕ_1, for sample sizes T = 50, 100, 200 and 500 the values of Thiel's U for positive values of ϕ_1 are more than the values of Thiel's U for negative values of ϕ_1, for all sample sizes the values of MSE are descending with the descending values of ϕ_1, for sample sizes, T = 30 the values of Thiel's U are ascending with the descending values of ϕ_1 and for sample sizes T = 50, 100, 200 and 500 the values of Thiel's U are descending with the descending values of ϕ_1.

CASE 3: The Second Condition of Stationary Conditions is Exists

In this case, the initial values for ρ_1 and ρ_2 are $\rho_1 = 2.3$ and $\rho_2 = 1.2$ which make the second condition of stationarity conditions for AR (2) model in equation (1) exists and for unbounded time series, it can be noticed from the table (4) in the appendix of tables that for all sample sizes the values of MSE for positive values of ϕ_1 are more than the values of MSE for negative values of ϕ_1, for all sample sizes the values of Thiel's U for positive values of ϕ_1 are more than the values of Thiel's U for negative values of ϕ_1, for sample sizes, T = 30 and 50 for positive values of ϕ_1 the values of MSE fluctuated with the descending values of ϕ_1, for sample sizes T = 30 and 50 for negative values of ϕ_1 the values of MSE are descending with the descending values of ϕ_1, for sample sizes, T = 100, 200 and 500 the values of MSE are descending with the descending values of ϕ_1 and in general for all sample sizes the values of Thiel's U are descending with the descending values of ϕ_1.

CASE 4: The First and the Second Conditions of Stationary Conditions are Exists

In this case, the initial values for ρ_1 and ρ_2 are $\rho_1 = 1.3$ and $\rho_2 = -1.5$ which make the first and the second conditions of stationarity conditions for AR (2) model in equation (1) are exists and for unbounded time series, it can be noticed from the table (5) in the appendix of tables that for all sample sizes the values of MSE and Thiel's U for positive values of ϕ_1 are more than the values of MSE and Thiel's U for negative values of ϕ_1 and for all sample sizes the values of MSE and Thiel's U are descending with the descending values of ϕ_1.

CASE 5: The First and the Third Conditions of Stationary Conditions are Exists

In this case, the initial values for ρ_1 and ρ_2 are $\rho_1 = -1.7$ and $\rho_2 = 0.8$ which make the first and the third conditions of stationarity conditions for AR (2) model in equation (1) are exists and for unbounded time series, it can be noticed from the table (6) in the appendix of tables that for all sample sizes the values of MSE for positive values of ϕ_1 are more than the values of MSE for negative values of ϕ_1, for sample size T = 30 the values of Thiel's U for positive values of ϕ_1 are less than the values of Thiel's U for negative values of ϕ_1, for sample sizes T = 50, 100, 200 and 500 the values of Thiel's U for positive values of ϕ_1 are more than the values of Thiel's U for negative values of ϕ_1, for all sample sizes the values of MSE are descending with the descending values of ϕ_1, for sample size T = 30 the values of Thiel's U are ascending with the descending values of ϕ_1 and for sample sizes T = 50, 100, 200 and 500 the values of Thiel's U are descending with the descending values of ϕ_1.

CASE 6: The Second and the Third Conditions of Stationary Conditions are Exists

In this case, the initial values for ρ_1 and ρ_2 are $\rho_1 = 1.7$ and $\rho_2 = 0.1$ which make the second and the third conditions of stationarity conditions for AR (2) model in equation (1) are exists and for unbounded time series, it can be noticed from the table (7) in the appendix of tables that for all sample sizes the values of MSE and Thiel's U for positive values of ϕ_1 are less than the values of MSE and Thiel's U for negative values of ϕ_1, for sample size T = 30 and 50 the values of MSE are fluctuate with the descending values of ϕ_1, for sample size T = 100, 200 and 500 MSE are descending with the descending values of ϕ_1, for all sample size and positive values of ϕ_1 the values of Thiel's U are descending with the descending values of ϕ_1 and for all sample size and negative values of ϕ_1 the values of Thiel's U are ascending with the descending values of ϕ_1.

6. Conclusion

- The GLS and the ML estimators, the unbiased and the variance-covariance matrix for the GLS and the ML estimators of parameters of AR (2) model with constant in case of dependent errors have been derived.

- For unbounded stationary time series and by using the measurements of MSE and Thiel's U the best case among all cases of unbounded stationary time series which gives the smallest values of MSE and Thiel's U is case five when the sample size is T = 500.

- When all conditions of stationary conditions for the AR (2) model in equation (1) are non-exists the values of MSE fluctuate with the ascending values of sample size T and the values of Thiel's U are ascending with the ascending values sample size T.

- When the first condition of stationary conditions for AR (2) model in equation (1) exists the values of MSE and Thiel's U are ascending with the ascending values sample size T.

- When the second condition of stationary conditions for AR (2) model in equation (1) exists the values of MSE and Thiel's U are ascending with the ascending values sample size T.

- When the first and the second conditions of stationary conditions for AR (2) model in equation (1) exists the values of MSE and Thiel's U are ascending with the ascending values of sample size T.

- When the first and the third conditions of stationary conditions for the AR (2) model in equation (1) exists the values of MSE are descending when the sample size increase from T = 30 to T = 50 and then the values of MSE are ascending with the ascending values of sample size T and the values of Thiel's U are ascending with the ascending values sample size T.

- When the second and the third conditions of stationary conditions for AR (2) model in equation (1) exists the values of MSE for positive values of ϕ_1 are descending in most cases with the ascending values of sample size T, the values of MSE for negative values of ϕ_1 are ascending with the ascending values of sample size T and for all sample sizes the values of Thiel's U are descending with the ascending values sample size T.

- By using the measurement of MSE the best case among all cases of nonstationary which gives the smallest values of MSE is case four when the first and the second conditions of stationary conditions for AR (2) model in equation (1) exists.

- By using the measurement of Thiel's U the best case among all cases of nonstationary which gives the smallest values of Thiel's U is case six when the second and the third conditions of stationary conditions for AR (2) model in equation (1) exists.
- Finally by using the measurements of MSE and Thiel's U the results for unbounded stationary time series are best than all cases of unbounded nonstationary time series.

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Appendix of Tables

Results of Bias, MSE, Thiel's U for GLS and ML Estimators of AR (2) Model for Unbounded Stationary Time Series

Table (1)

	T = 30					T = 50				
ϕ_1	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U
0.5	-0.162	-0.426	0.298	1759.837	0.159	-0.393	0.262	-0.349	3326.305	0.112
0.4	-0.222	-0.653	0.477	3545.570	0.163	0.279	-0.824	0.811	5037.994	0.117
0.3	1.119	-2.239	2.546	724.385	0.170	-0.092	0.085	-0.180	16.671	0.122
0.2	-0.017	-0.323	0.166	614.416	0.179	-0.240	0.690	-0.799	4450.855	0.130
0.1	-0.135	0.044	-0.236	93.757	0.188	-0.059	0.006	-0.094	37.354	0.140
- 0.1	-0.121	-0.061	-0.092	32.165	0.213	-0.083	-0.044	-0.044	2.346	0.163
- 0.2	-0.116	-0.120	-0.043	29.371	0.228	-0.087	-0.034	-0.054	1.315	0.177
- 0.3	-0.005	-0.200	0.035	106.404	0.248	-0.082	-0.052	-0.036	1.331	0.194
- 0.4	-0.086	0.020	-0.186	40.620	0.270	-0.088	-0.046	-0.041	1.280	0.214
- 0.5	-0.079	-0.090	-0.067	19.521	0.297	-0.087	-0.055	-0.032	1.335	0.238

	T = 100					T = 200				
ϕ_1	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U
0.5	-0.123	0.266	-0.307	144.171	0.071	-0.100	0.017	-0.023	5.339	0.046
0.4	-0.116	0.091	-0.128	55.385	0.076	-0.096	0.063	-0.070	3.267	0.051
0.3	-0.171	0.159	-0.191	75.671	0.082	-0.102	0.009	-0.016	3.156	0.057
0.2	-0.092	0.018	-0.052	1.518	0.089	-0.100	0.017	-0.023	1.434	0.062
0.1	-0.093	0.006	-0.039	1.102	0.097	-0.100	-0.003	-0.003	1.210	0.069
- 0.1	-0.093	-0.016	-0.017	1.043	0.117	-0.098	-0.006	-0.0001	1.062	0.083
- 0.2	-0.093	-0.020	-0.012	1.067	0.128	-0.098	-0.009	0.003	1.083	0.091
- 0.3	-0.093	-0.024	-0.009	1.117	0.142	-0.098	-0.011	0.004	1.132	0.101
- 0.4	-0.093	-0.026	-0.007	1.202	0.157	-0.098	-0.012	0.006	1.217	0.113
- 0.5	-0.093	-0.027	-0.006	1.337	0.176	-0.098	-0.013	0.007	1.351	0.127

	T = 500				
ϕ_1	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U
0.5	-0.100	0.003	0.007	4.171	0.030
0.4	-0.097	0.014	-0.005	1.481	0.033
0.3	-0.097	0.006	0.003	1.309	0.036
0.2	-0.097	0.002	0.008	1.206	0.040
0.1	-0.097	-0.001	0.010	1.143	0.044
- 0.1	-0.098	-0.004	0.013	1.101	0.053
- 0.2	-0.098	-0.005	0.014	1.118	0.059
- 0.3	-0.098	-0.005	0.015	1.163	0.065
- 0.4	-0.0979	-0.006	0.015	1.244	0.072
- 0.5	-0.0978	-0.006	0.015	1.376	0.082

Results of Bias, MSE, Thiel's U for GLS and ML Estimators of AR (2) Model Unbounded Nonstationary Time Series

Table (2)

	T = 30					T = 50				
ϕ_1	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U
0.5	-0.3930	-0.2840	-2.3140	948.1200	0.5020	-0.3639	-0.0289	-2.5501	1215.3044	0.5208
0.4	-0.6761	-1.0008	-1.5711	9624.774	0.5013	0.5389	-0.0306	-2.5418	1516.6306	0.5213
0.3	1.1390	-1.3361	-1.1282	12499.37	0.5023	-0.5073	2.0572	-4.6908	45514.163	0.5209
0.2	0.3485	-0.3292	-2.2769	1919.046	0.5029	-0.6745	-1.6600	-0.8308	10549.906	0.5220
0.1	-0.1511	0.0011	-2.6021	447.7445	0.5045	-0.5379	-0.4492	-2.1022	1348.4792	0.5225
- 0.1	-0.1289	-0.1691	-2.4350	455.9769	0.5088	-0.1059	0.1901	-2.7521	232.9151	0.5251
- 0.2	-0.2416	0.0256	-2.6290	235.6406	0.5126	3.2589	-6.4286	4.0261	182656.80	0.5276
- 0.3	-0.0704	-0.0897	-2.5162	93.1722	0.5182	-0.0685	-0.2280	-2.3307	192.5253	0.5310
- 0.4	-0.2977	-0.1900	-2.4213	244.0119	0.5256	-0.1015	-0.0797	-2.4819	78.8887	0.5358
- 0.5	-0.0894	-0.0706	-2.5371	50.7405	0.5355	-0.1059	-0.1039	-2.4581	66.6973	0.5424
	T = 100					T = 200				
ϕ_1	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U
0.5	0.2121	-2.0799	-0.4222	8335.822	0.5374	-0.0449	-0.1116	-2.4038	2487.4137	0.5461
0.4	-0.2192	-0.1568	-2.3651	2645.861	0.5375	-0.5239	0.1514	-2.6626	2826.6582	0.5460
0.3	-0.0391	0.1103	-2.6395	702.2164	0.5373	-0.1802	-0.3547	-2.1569	1486.7336	0.5460
0.2	0.2535	0.5574	-3.0919	2804.786	0.5370	-0.0927	-0.0443	-2.4714	949.8193	0.5458
0.1	-0.0520	-0.2773	-2.2500	608.8197	0.5374	-0.1111	-0.0945	-2.4210	752.5519	0.5460
- 0.1	-0.1033	-0.0849	-2.4454	246.8071	0.5389	-0.0923	-0.0776	-2.4378	505.6318	0.5468
- 0.2	-0.0923	-0.0867	-2.4434	207.9054	0.5401	-0.0954	-0.0895	-2.4259	425.2468	0.5474
- 0.3	-0.0956	-0.0914	-2.4389	177.9503	0.5420	-0.0961	-0.0941	-2.4212	363.2696	0.5484
- 0.4	-0.0959	-0.1045	-2.4256	154.1063	0.5447	-0.0965	-0.0971	-2.4182	314.3058	0.5498
- 0.5	-0.0959	-0.1105	-2.4196	135.4833	0.5486	-0.0967	-0.0992	-2.4161	275.1071	0.5519
	T = 500									
ϕ_1	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U					
0.5	-0.1423	-0.0890	-2.4173	5993.436	0.5515					
0.4	-0.1027	-0.0648	-2.4414	4168.803	0.5515					
0.3	-0.1093	-0.0758	-2.4304	3066.791	0.5515					
0.2	-0.1083	-0.0853	-2.4208	2350.387	0.5516					
0.1	-0.1075	-0.0906	-2.4155	1858.778	0.5516					
- 0.1	-0.1062	-0.0964	-2.4096	1246.458	0.5520					
- 0.2	-0.1057	-0.0981	-2.4080	1048.376	0.5523					
- 0.3	-0.1053	-0.0994	-2.4067	894.3062	0.5527					
- 0.4	-0.1049	-0.1003	-2.4058	772.2282	0.5532					
- 0.5	-0.1046	-0.1009	-2.4052	674.0333	0.5541					

Table (3)

	T = 30					T = 50				
ϕ_1	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U
0.5	0.0024	2.1324	-0.8396	230.241	0.8169	-0.0014	2.2481	-0.7783	233.3704	0.9108
0.4	-0.0989	2.1347	-0.8378	202.336	0.8226	-0.0484	2.2379	-0.7675	154.8580	0.9118
0.3	-0.0508	2.1458	-0.8489	62.727	0.8282	-0.0518	2.2381	-0.7687	115.8560	0.9118
0.2	-0.0923	2.2274	-0.9432	138.731	0.8322	-0.0515	2.2324	-0.7633	90.2535	0.9112
0.1	-0.0435	2.1669	-0.8712	40.562	0.8354	-0.0514	2.2338	-0.7658	72.5826	0.9102
- 0.1	-0.0487	2.1543	-0.8573	27.879	0.8410	-0.0520	2.2303	-0.7639	50.6517	0.9073
- 0.2	-0.0472	2.1540	-0.8586	24.604	0.8435	-0.0520	2.2303	-0.7649	43.7310	0.9054
- 0.3	-0.0448	2.1412	-0.8445	23.344	0.8462	-0.0521	2.2308	-0.7665	38.5427	0.9033
- 0.4	-0.0455	2.1567	-0.8653	20.693	0.8491	-0.0521	2.2316	-0.7685	34.7117	0.9010
- 0.5	-0.0445	2.1578	-0.8689	19.858	0.8525	-0.0521	2.2327	-0.7708	32.0495	0.8986

	T = 100					T = 200				
ϕ_1	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U
0.5	-0.0522	2.3201	-0.6713	539.3037	0.9796	-0.0740	2.3734	-0.6137	1232.694	0.9955
0.4	-0.0557	2.3166	-0.6686	378.3512	0.9779	-0.0758	2.3719	-0.6126	860.5195	0.9943
0.3	-0.0580	2.3147	-0.6674	280.3207	0.9758	-0.0771	2.3710	-0.6121	634.9675	0.9928
0.2	-0.0598	2.3136	-0.6670	216.3193	0.9734	-0.0781	2.3705	-0.6119	488.0982	0.9910
0.1	-0.0612	2.3130	-0.6670	172.3054	0.9704	-0.0788	2.3703	-0.6119	387.2087	0.9888
- 0.1	-0.0633	2.3127	-0.6677	117.5608	0.9630	-0.0798	2.3702	-0.6121	261.5877	0.9828
- 0.2	-0.0641	2.3129	-0.6683	100.0006	0.9583	-0.0802	2.3703	-0.6124	221.0740	0.9789
- 0.3	-0.0648	2.3132	-0.6691	86.5219	0.9530	-0.0806	2.3705	-0.6128	189.7184	0.9741
- 0.4	-0.0654	2.3137	-0.6700	76.1053	0.9467	-0.0809	2.3708	-0.6132	165.1052	0.9683
- 0.5	-0.0659	2.3144	-0.6712	68.1261	0.9395	-0.0811	2.3711	-0.6137	145.6613	0.9611

	T = 500				
ϕ_1	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U
0.5	-0.0686	2.4085	-0.5696	3287.694	0.9987
0.4	-0.0726	2.4080	-0.5693	2288.498	0.9982
0.3	-0.0754	2.4078	-0.5691	1684.625	0.9975
0.2	-0.0776	2.4077	-0.5690	1292.072	0.9966
0.1	-0.0793	2.4076	-0.5690	1022.664	0.9955
- 0.1	-0.0818	2.4076	-0.5691	687.2055	0.9925
- 0.2	-0.0828	2.4077	-0.5692	578.7912	0.9903
- 0.3	-0.0836	2.4078	-0.5693	494.5810	0.9875
- 0.4	-0.0843	2.4079	-0.5695	428.0222	0.9839
- 0.5	-0.0849	2.4080	-0.5697	374.7355	0.9792

Table (4)

	T = 30					T = 50				
ϕ_1	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U
0.5	-3.4312	-2.0387	-0.6101	52310.334	0.5103	0.1715	-0.7778	-1.7753	1309.404	0.5264
0.4	0.3026	-0.5168	-2.1562	4185.2382	0.5094	-2.8526	4.2700	-7.1237	98974.06	0.5263
0.3	0.3382	-1.0543	-1.5413	1815.9965	0.5091	0.1540	-0.6343	-1.9353	756.939	0.5258
0.2	-0.8598	-0.1127	-2.5132	3351.6985	0.5099	0.0618	-1.6469	-0.8783	2653.297	0.5260
0.1	-0.1705	-1.5743	-0.9492	3470.1618	0.5083	-0.2321	-0.6346	-1.9244	489.081	0.5237
- 0.1	-0.0286	-0.5571	-2.0676	141.2536	0.5074	-0.0920	-0.5358	-2.0315	131.221	0.5222
- 0.2	-0.0433	-0.4830	-2.1322	80.5986	0.5068	-0.0926	-0.5892	-1.9749	104.950	0.5218
- 0.3	-0.0702	-0.6116	-2.0029	85.2520	0.5074	-0.0982	-0.6780	-1.8789	145.048	0.5222
- 0.4	-0.0826	-0.6268	-1.9837	51.5509	0.5087	-0.0976	-0.6260	-1.9371	77.477	0.5230
- 0.5	-0.0810	-0.6177	-1.9914	44.4388	0.5122	-0.0961	-0.6406	-1.9221	68.153	0.5251
	T = 100					**T = 200**				
ϕ_1	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U
0.5	0.1785	-0.4881	-2.0521	2128.389	0.5411	-0.3692	-0.8298	-1.6858	2710.045	0.5482
0.4	0.1540	-0.3457	-2.1917	2024.014	0.5404	-0.0530	-0.5453	-1.9711	1795.078	0.5478
0.3	0.2050	-0.4534	-2.0786	1045.529	0.5400	-0.1177	-0.4910	-2.0249	1263.454	0.5475
0.2	-0.0423	-0.5265	-2.0057	487.194	0.5392	-0.0964	-0.5274	-1.9884	963.029	0.5470
0.1	-0.0794	-0.4853	-2.0478	388.224	0.5386	-0.0993	-0.5704	-1.9453	761.247	0.5466
- 0.1	-0.1065	-0.5826	-1.9487	251.529	0.5376	-0.0998	-0.5995	-1.9160	510.920	0.5462
- 0.2	-0.1023	-0.6019	-1.9291	211.630	0.5374	-0.0997	-0.6061	-1.9093	429.966	0.5461
- 0.3	-0.1011	-0.6127	-1.9182	180.867	0.5375	-0.0997	-0.6105	-1.9049	367.006	0.5461
- 0.4	-0.1003	-0.6197	-1.9111	156.594	0.5379	-0.0996	-0.6134	-1.9020	317.140	0.5462
- 0.5	-0.0998	-0.6244	-1.9063	137.191	0.5388	-0.0995	-0.6154	-1.9000	277.070	0.5466
	T = 500									
ϕ_1	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U					
0.5	-0.2784	-0.2053	-2.3011	6490.813	0.5526					
0.4	-0.1004	-0.5335	-1.9727	4194.880	0.5523					
0.3	-0.1039	-0.5692	-1.9370	3084.210	0.5521					
0.2	-0.1039	-0.5865	-1.9197	2363.296	0.5520					
0.1	-0.1039	-0.5959	-1.9103	1868.674	0.5519					
- 0.1	-0.1036	-0.6055	-1.9006	1252.574	0.5517					
- 0.2	-0.1034	-0.6081	-1.8981	1053.210	0.5517					
- 0.3	-0.1032	-0.6098	-1.8963	898.078	0.5517					
- 0.4	-0.1030	-0.6111	-1.8950	775.065	0.5517					
- 0.5	-0.1028	-0.6120	-1.8941	675.981	0.5518					

Table (5)

	T = 30					T = 50				
ϕ_1	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U
0.5	-0.0288	-0.6668	1.0724	21.721	0.6990	-0.0327	-0.6519	1.1622	35.704	0.7788
0.4	-0.0296	-0.6852	1.1087	19.671	0.6990	-0.0325	-0.6532	1.1691	25.783	0.7733
0.3	-0.0297	-0.6634	1.0925	11.795	0.6984	-0.0330	-0.6565	1.1772	19.882	0.7677
0.2	-0.0296	-0.6729	1.1121	9.146	0.6969	-0.0334	-0.6585	1.1829	15.868	0.7619
0.1	-0.0298	-0.6770	1.1234	7.760	0.6944	-0.0337	-0.6598	1.1872	13.007	0.7559
- 0.1	-0.0298	-0.6766	1.1316	5.700	0.6864	-0.0340	-0.6613	1.1929	9.278	0.7427
- 0.2	-0.0299	-0.6777	1.1361	5.018	0.6808	-0.0341	-0.6618	1.1948	8.019	0.7352
- 0.3	-0.0299	-0.6784	1.1393	4.465	0.6737	-0.0342	-0.6622	1.1963	7.014	0.7266
- 0.4	-0.0299	-0.6791	1.1418	4.009	0.6649	-0.0343	-0.6626	1.1974	6.200	0.7166
- 0.5	-0.0298	-0.6799	1.1437	3.630	0.6537	-0.0343	-0.6631	1.1983	5.530	0.7043

	T = 100					T = 200				
ϕ_1	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U
0.5	-0.0407	-0.6213	1.2393	93.624	0.8662	-0.0443	-0.5766	1.2975	248.760	0.9212
0.4	-0.0412	-0.6229	1.2425	66.452	0.8559	-0.0458	-0.5777	1.2989	174.158	0.9124
0.3	-0.0416	-0.6239	1.2449	49.733	0.8467	-0.0468	-0.5784	1.2999	128.858	0.9043
0.2	-0.0418	-0.6246	1.2467	38.704	0.8382	-0.0476	-0.5789	1.3007	99.284	0.8967
0.1	-0.0420	-0.6250	1.2480	31.038	0.8301	-0.0482	-0.5792	1.3012	78.906	0.8896
- 0.1	-0.0422	-0.6255	1.2497	21.336	0.8144	-0.0491	-0.5796	1.3019	53.385	0.8759
- 0.2	-0.0423	-0.6255	1.2502	18.147	0.8062	-0.0494	-0.5797	1.3022	45.079	0.8691
- 0.3	-0.0423	-0.6256	1.2505	15.641	0.7975	-0.0497	-0.5798	1.3023	38.592	0.8619
- 0.4	-0.0424	-0.6256	1.2508	13.637	0.7877	-0.0499	-0.5799	1.3025	33.427	0.8541
- 0.5	-0.0424	-0.6257	1.2509	12.007	0.7761	-0.0501	-0.5800	1.3026	29.247	0.8452

	T = 500				
ϕ_1	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U
0.5	-0.0348	-0.5115	1.3724	846.788	0.9648
0.4	-0.0400	-0.5119	1.3728	589.519	0.9595
0.3	-0.0437	-0.5122	1.3730	434.062	0.9545
0.2	-0.0465	-0.5124	1.3732	332.988	0.9497
0.1	-0.0486	-0.5126	1.3734	263.585	0.9451
- 0.1	-0.0518	-0.5128	1.3736	177.047	0.9361
- 0.2	-0.0530	-0.5129	1.3736	149.004	0.9316
- 0.3	-0.0540	-0.5129	1.3737	127.156	0.9269
- 0.4	-0.0549	-0.5130	1.3737	109.803	0.9220
- 0.5	-0.0556	-0.5131	1.3738	95.789	0.9165

Table (6)

	T = 30					T = 50				
ϕ_1	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U
0.5	0.0030	2.0010	-0.6701	364.5647	0.8501	-0.0658	1.9846	-0.4638	239.1173	0.9350
0.4	-0.0815	1.8938	-0.5407	108.3336	0.8563	-0.0602	1.9941	-0.4749	163.1285	0.9355
0.3	-0.0469	1.9321	-0.5825	68.9981	0.8610	-0.0556	1.9815	-0.4612	121.9595	0.9353
0.2	-0.0605	1.9074	-0.5495	51.2894	0.8642	-0.0564	1.9785	-0.4587	94.8081	0.9344
0.1	-0.0520	1.9255	-0.5720	42.2693	0.8664	-0.0567	1.9743	-0.4550	76.0865	0.9329
- 0.1	-0.0531	1.9045	-0.5476	28.9414	0.8692	-0.0569	1.9744	-0.4568	52.7069	0.9286
- 0.2	-0.0353	1.9369	-0.5922	38.8925	0.8701	-0.0569	1.9742	-0.4575	45.2562	0.9258
- 0.3	-0.0510	1.9023	-0.5480	22.5933	0.8707	-0.0570	1.9746	-0.4589	39.6036	0.9226
- 0.4	-0.0504	1.9072	-0.5554	20.7098	0.8716	-0.0570	1.9753	-0.4606	35.3366	0.9190
- 0.5	-0.0495	1.9066	-0.5568	19.4730	0.8726	-0.0570	1.9764	-0.4628	32.2267	0.9150

	T = 100					T = 200				
ϕ_1	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U
0.5	-0.0584	2.0577	-0.3744	557.7909	0.9870	-0.0778	2.1032	-0.3222	1257.905	0.9971
0.4	-0.0615	2.0531	-0.3703	391.3944	0.9856	-0.0794	2.1011	-0.3204	878.1389	0.9962
0.3	-0.0635	2.0508	-0.3686	289.9139	0.9839	-0.0805	2.1000	-0.3196	647.8836	0.9950
0.2	-0.0650	2.0495	-0.3678	223.5944	0.9817	-0.0813	2.0994	-0.3192	497.8976	0.9936
0.1	-0.0662	2.0488	-0.3676	177.9401	0.9791	-0.0819	2.0991	-0.3191	394.8272	0.9918
- 0.1	-0.0680	2.0483	-0.3680	121.0438	0.9724	-0.0828	2.0989	-0.3192	266.3912	0.9869
- 0.2	-0.0687	2.0485	-0.3685	102.7283	0.9680	-0.0832	2.0990	-0.3195	224.9107	0.9835
- 0.3	-0.0693	2.0488	-0.3692	88.6108	0.9628	-0.0834	2.0991	-0.3198	192.7532	0.9794
- 0.4	-0.0698	2.0493	-0.3701	77.6204	0.9567	-0.0837	2.0994	-0.3202	167.4383	0.9743
- 0.5	-0.0702	2.0499	-0.3712	69.0822	0.9495	-0.0839	2.0998	-0.3207	147.3339	0.9679

	T = 500				
ϕ_1	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U
0.5	-0.0730	2.1315	-0.2827	3317.624	0.9991
0.4	-0.0765	2.1310	-0.2822	2309.237	0.9987
0.3	-0.0789	2.1307	-0.2820	1699.758	0.9982
0.2	-0.0808	2.1305	-0.2819	1303.527	0.9975
0.1	-0.0823	2.1304	-0.2818	1031.564	0.9967
- 0.1	-0.0845	2.1304	-0.2819	692.8407	0.9943
- 0.2	-0.0853	2.1305	-0.2820	583.3182	0.9925
- 0.3	-0.0860	2.1305	-0.2821	498.1975	0.9903
- 0.4	-0.0866	2.1307	-0.2822	430.8515	0.9874
- 0.5	-0.0871	2.1308	-0.2824	376.8355	0.9834

Table (7)

ϕ_1	T = 30 bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U	T = 50 bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U
0.5	-4.7120	9.4870	-10.5890	5.62652.	0.3080	-0.0509	-1.0110	0.1533	2407.577	0.2988
0.4	-0.1759	-0.6813	-0.1348	2893.669	0.3052	0.0426	-0.4466	-0.4269	638.176	0.2947
0.3	-0.0886	-0.2880	-0.6536	231.0125	0.3009	-0.2195	0.1964	-1.0888	120.148	0.2897
0.2	-0.2344	0.1468	-1.0912	244.2930	0.2987	0.0335	-0.1354	-0.7375	151.031	0.2879
0.1	-0.0561	-0.7915	-0.1193	2036.596	0.2958	-0.0484	-0.1112	-0.7689	39.295	0.2865
- 0.1	-1.7422	1.0786	-1.4115	873.6613	0.2970	-0.1063	-0.1077	-0.7725	20.744	0.2890
- 0.2	-0.0427	-0.1662	-0.7772	20.6669	0.3006	-0.1937	-0.3566	-0.5224	205.351	0.2922
- 0.3	-0.0850	-0.1686	-0.7650	19.2119	0.3058	-0.1115	-0.1805	-0.6972	15.132	0.2972
- 0.4	-0.0717	-0.1939	-0.7400	10.8095	0.3148	-0.0909	-0.2259	-0.6510	9.326	0.3045
- 0.5	-0.0820	-0.2658	-0.6662	6.2102	0.3270	-0.0938	-0.2375	-0.6393	8.205	0.3145

ϕ_1	T = 100 bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U	T = 200 bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U
0.5	-0.1013	-0.3250	-0.5198	1007.701	0.2891	0.2559	0.2215	-1.0344	1558.600	0.2865
0.4	-0.0668	-0.0326	-0.8038	244.189	0.2868	-0.1228	-0.0989	-0.7195	285.099	0.2854
0.3	-0.0761	-0.0915	-0.7500	164.488	0.2854	-0.1166	-0.0647	-0.7548	134.100	0.2843
0.2	-0.1122	-0.1805	-0.6599	94.615	0.2853	-0.0997	-0.1516	-0.6676	99.378	0.2842
0.1	-0.0916	-0.1094	-0.7300	40.327	0.2845	-0.1015	-0.1575	-0.6618	78.776	0.2846
- 0.1	-0.0968	-0.1649	-0.6736	26.865	0.2866	-0.0988	-0.1859	-0.6331	53.025	0.2860
- 0.2	-0.0973	-0.1884	-0.6497	22.500	0.2888	-0.0986	-0.1916	-0.6274	44.791	0.2872
- 0.3	-0.0973	-0.1996	-0.6384	19.411	0.2920	-0.0984	-0.1955	-0.6235	38.431	0.2890
- 0.4	-0.0973	-0.2069	-0.6310	17.042	0.2965	-0.0983	-0.1981	-0.6208	33.452	0.2915
- 0.5	-0.0973	-0.2120	-0.6258	15.236	0.3029	-0.0983	-0.2000	-0.6189	29.535	0.2952

ϕ_1	T = 500 bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U
0.5	-0.0966	-0.1185	-0.6892	650.965	0.2853
0.4	-0.1024	-0.1227	-0.6850	431.069	0.2850
0.3	-0.1031	-0.1590	-0.6486	316.506	0.2850
0.2	-0.1030	-0.1721	-0.6355	242.632	0.2850
0.1	-0.1027	-0.1796	-0.6280	191.968	0.2852
- 0.1	-0.1023	-0.1875	-0.6200	128.932	0.2858
- 0.2	-0.1021	-0.1897	-0.6178	108.581	0.2863
- 0.3	-0.1019	-0.1912	-0.6163	92.789	0.2871
- 0.4	-0.1017	-0.1923	-0.6152	80.325	0.2882
- 0.5	-0.1016	-0.1931	-0.6144	70.370	0.2898