H^N- ENTROPY: A NEW MEASUREOF INFORMATION AND ITS PROPERTIES

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ABSTRACT

Entropy measures the amount of uncertainty and dispersion of an unknown or random quantity, this concept introduced at first by Shannon (1948), it is important for studies in many areas. Like, information theory: entropy measures the amount of information in each message received, physics: entropy is the basic concept that measures the disorder of the thermodynamical system, and others. Then, in this paper, we introduce an alternative measure of entropy, called H^N- entropy, unlike Shannon entropy, this proposed measure of order α and β is more flexible than Shannon. Then, the cumulative residual H^N- entropy, cumulative H^N- entropy, and weighted version have been introduced. Finally, comparison between Shannon entropy and H^N- entropy and numerical results have been introduced.

KEYWORDS

Shannon entropy. Rényi entropy. Tsallis entropy. Varma entropy. Weighted entropy. Cumulative (residual) entropy. H^N- entropy. IRL, DRL (Increasing) Decreasing Relative loss entropy.

1. INTRODUCTION AND BACKGROUND

Let X be a continuous non-negative absolutely continuous random variable having probability density function (PDF) $f(x)$, survival function $\bar{F}(x) = 1 - F(x)$, and cumulative distribution function (CDF) $F(x)$. Then, the classical measure of uncertainty for X is the differential entropy, also known as the Shannon information measure, defined as:

$$H_{sh}(X) = -E(\ln f(x))$$
$$= -\int_0^\infty f(x)\ \ln f(x)\, dx, \tag{1}$$

where ln denotes the natural logarithm. Since the classical contributions by Shannon [18], the properties have been thoroughly investigated.

Furthermore, many generalizations of (1) have been proposed. For instance, Rényi [16] introduced another entropy measure of order α, defined as:

$$I_R(X) = \frac{1}{1-\alpha} ln\left[\int_0^\infty f^\alpha(x)\,dx\right],\alpha \neq 1,\alpha > 0, \tag{2}$$

where $I_R(1) = \lim_{\alpha \to 1} I_R(\alpha) = H_{sh}(X)$.

After this, another entropy measure of order α and β introduced by Varma [20], as:

$$V_{\alpha,\beta}(X) = \frac{1}{\beta-\alpha} ln\left[\int_0^\infty f^{\alpha+\beta-1}(x)\,dx\right];\ (\beta-1) < \alpha < \beta,\beta > 1. \tag{3}$$

Then, Tsallis [19] introduced another entropy measure of order α, defined as:

$$T_\alpha(X) = \frac{1}{\alpha-1}\left[1 - \int_0^\infty f^\alpha(x)\,dx\right],\alpha \neq 1,\alpha > 0. \tag{4}$$

A new measure introduced by Rao et al. [14] as a measure of uncertainty related to the future, this measure called cumulative residual entropy defined as:

$$\varepsilon(X) = -\int_0^\infty \bar{F}(x)\ ln\,\bar{F}(x)\ dx. \tag{5}$$

Then, Di Crescenzo and Longobardi [5] introduced another information measure similar to $\varepsilon(X)$. That turns out to be particularly useful to measure information on inactivity time of a system, given by:

$$C\varepsilon(X) = -\int_0^\infty F(x)\ ln\,F(x)\,dx. \tag{6}$$

The arrangement of the current paper is provided as follows: In Section 2, we introduced a new measure of order α and β, called H^N- entropy, the cumulative residual H^N- entropy, and the cumulative H^N- entropy. Section 3, the weighted version of this new measures by using length – biased weighted function have been introduced. Section4, stochastic orders based on H^N- entropy. Section 5, a comparison between Shannon entropy and new entropy and numerical results have been derived.

derived.

2. H^N- ENTROPY

In analogy with (1), we define the H^N- entropy as a new measure of uncertainty with parameters (α,β) of a non – negative random variable X , as:

$$H^N(X) = E\left(g(f(x))\right)$$

$$= \frac{1}{\alpha+\beta}\left(1 - \int_0^\infty f(x)\ \ ln\,f^{\alpha+\beta}(x)\ \ dx\right);\ \ \alpha > 0\,,\beta > 0, \tag{7}$$

where $g(f(x))$ is a convex function, written as:

$$g(f(x)) = \frac{1}{\alpha+\beta}\left(1 - ln\,f^{\alpha+\beta}(x)\right).$$

Proposition 2.1. Let X and Y be two non - negative random variables, and if $y = ax + b$, with $a > 0$, and $b > 0$, then

$$H^N(Y) = H^N(X) + ln|a|.$$

Proof:

Let $y = ax + b$, then $x = \frac{y-b}{a}$, $f_y(y) = \frac{1}{a} f_x(\frac{y-b}{a})$

$$H^N(Y) = \frac{1}{\alpha+\beta}\left[1 - \int_b^{\infty} f_y(y)\ ln\, f_y{}^{\alpha+\beta}(y) \quad dy\right] = H^N(X) + ln|a|.$$

Proposition 2.2. We recall that the two – dimensional version of (7), defined as:

$$H^N(X,Y) = E\left(g\left(f(x,y)\right)\right)$$

$$= \frac{1}{\alpha+\beta}\left(1 - \int_0^{\infty}\int_0^{\infty} f(x,y)\ \ ln\, f^{\alpha+\beta}(x,y)\ \ dy\, dx\ \right);\quad \alpha > 0\ ,\beta > 0, \qquad (8)$$

has the following property: if X and Y are independent, then:

$$H^N(X,Y) = H^N(X) + H^N(Y) - \frac{1}{\alpha+\beta}.$$

Proof:

If X ,Y are independent, then the $f(x,y) = f(x)f(y)$, then

$$H^N(X\ ,Y) = \frac{1}{\alpha+\beta}\left[1 - \int_0^{\infty}\int_0^{\infty} f(x)f(y)\, ln\left(f^{\alpha+\beta}(x)f^{\alpha+\beta}(y)\right)\, dydx\right]$$

$$= H^N(X) + H^N(Y) - \frac{1}{\alpha+\beta}.$$

Then, we introduce another entropy measure based on H^N – entropy, this new measure has been introduced as a measure of uncertainty when the uncertainty related to the future lifetime of a system, called the cumulative residual H^N – entropy measure. The basic concept of this measure is to replace the PDF with the survival function, where is the CDF is more regular than PDF, because it is computed as the derivative of CDF, defined as:

$$\varepsilon H^N(X) = \frac{1}{\alpha+\beta}\left[1 - \int_0^{\infty} \bar{F}(x)\, ln\, \bar{F}^{\alpha+\beta}(x)dx\right], \alpha > 0, \beta > 0. \qquad (9)$$

This new measure has many important properties unlike the differential entropy such as: 1) Cumulative residual entropy has consistent definition in both the continuous and discrete domains. 2) Cumulative residual entropy is always non – negative. 3) Cumulative residual entropy can be easily computed from sample data and these computations asymptotically converge to the true values.

Proposition 2.3. According to proof of proposition 2.1, then

$$\varepsilon H^N(Y) = \varepsilon H^N(X) + ln|a|.$$

Proposition 2.4. We recall that the two – dimensional version of (9), defined as:

$$\varepsilon H^N(X,Y) = \frac{1}{\alpha+\beta}\left(1 - \int_0^\infty \int_0^\infty \bar{F}(x,y) \;\; ln\, \bar{F}^{\alpha+\beta}(x,y) \;\; dy\, dx \;\right); \quad \alpha > 0\,, \beta > 0, \qquad (10)$$

if X and Y are independent, then, as proposition 2.2, we have

$$\varepsilon H^N(X,Y) = E(y)\varepsilon H^N(X) + E(x)\varepsilon H^N(Y) + \frac{1 - E(y) - E(x)}{\alpha+\beta},$$

where $E(x) = \int_0^\infty \bar{F}(x)dx$, and $E(y) = \int_0^\infty \bar{F}(y)dy$, are the expectation of X and Y.

Then, we define the cumulative H^N – entropy of a non – negative random variable X, as:

$$C\varepsilon H^N(X) = \frac{1}{\alpha+\beta}\left[1 - \int_0^\infty F(x)\, ln\, F^{\alpha+\beta}\,(x)dx\right], \alpha > 0, \beta > 0. \qquad (11)$$

In particular $C\varepsilon H^N(X) = 0$ if only if X is constant, and the basic concept of this measure is suitable to describe the true elapsing between the failure system and the time when it is found to be down.

Remark 2.1. Let $\mu = E(x)$ be finite, then the cumulative residual H^N-entropy is equal to the cumulative H^N-entropy if the distribution is symmetric with respect to μ, i.e. if $F(\mu + x) = 1 - F(\mu - x)$.

Proposition 2.5. If $Y = aX + b$, with $a > 0$ and $b > 0$, then

$$C\varepsilon H^N(Y) = C\varepsilon H^N(X) + ln|a|.$$

Proposition 2.6. We remember that the two – dimensional version of (11), defined as:

$$C\varepsilon H^N(X,Y) = \frac{1}{\alpha+\beta}\left(1 - \int_0^\infty \int_0^\infty F(x,y) \;\; ln\, F^{\alpha+\beta}(x,y) \;\; dy\, dx \;\right); \quad \alpha > 0\,, \beta > 0, \qquad (12)$$

if X and Y are independent, then, as proposition 2.2, we have

$$C\varepsilon H^N(X,Y) = \delta(y)C\varepsilon H^N(X) + \delta(x)C\varepsilon H^N(Y) + \frac{1 - \delta(y) - \delta(x)}{\alpha+\beta},$$

where $\delta(x) = \int_0^\infty F(x)dx$, and $\delta(y) = \int_0^\infty F(y)dy$.

3. WEIGHTED H^N - ENTROPY

At times, elementary events do not have the same important, so it is needful to associate the probability and qualitative weights. For that, the new entropy measures have been introduced with weighted function, for instance, Di Crescenzo and Longobardi [4] introduced the weighted entropy, Misagh et al. [11] introduced the weighted cumulative entropy, Mirali et al. [10] introduced the weighted cumulative residual entropy, and Nourbakhsh and Yari [13] introduced the weighted Rényi entropy.

Then, we introduced another H^N- entropy measure with length – biased weighted function given as:

$$H_w^N(X) = \frac{1}{\alpha+\beta}\left(1 - \int_0^\infty x\, f(x)\ \ln f^{\alpha+\beta}(x)\quad dx\ \right);\quad \alpha > 0\ ,\beta > 0. \qquad (13)$$

Proposition 3.1. We recall that the two – dimensional version of (13), defined as :

$$H_w^N(X,Y) = \frac{1}{\alpha+\beta}\left(1 - \int_0^\infty \int_0^\infty xyf(x,y)\ \ln f^{\alpha+\beta}(x,y)\quad dy\, dx\ \right);\quad \alpha > 0\ ,\beta > 0, \quad (14)$$

if X and Y are independent, then:

$$H_w^N(X,Y) = E(y)H_w^N(X) + E(X)H_w^N(Y) + \frac{1 - E(y) - E(x)}{\alpha + \beta}.$$

Proof:

$$H_w^N(X,Y) = \frac{1}{\alpha+\beta}\left[1 - \left(E(y)\int_0^\infty xf(x)\ln f^{\alpha+\beta}(x)dx + E(x)\int_0^\infty yf(y)\ln f^{\alpha+\beta}(y)dy\right)\right]$$

$$= E(y)H_w^N(X) + E(X)H_w^N(Y) + \frac{1-E(y)-E(x)}{\alpha+\beta}.$$

Then, we introduce the weighted cumulative residual H^N – entropy with weighted function $w(x) = x$, as:

$$\varepsilon H_w^N(X) = \frac{1}{\alpha+\beta}\left[1 - \int_0^\infty x\bar{F}(x)\ln \bar{F}^{\alpha+\beta}\ (x)dx\right], \alpha > 0, \beta > 0. \qquad (15)$$

Proposition 3.2. We remember that the two – dimensional version of (15), defined as:

$$\varepsilon H_w^N(X,Y) = \frac{1}{\alpha+\beta}\left(1 - \int_0^\infty \int_0^\infty xy\,\bar{F}(x,y)\ \ln \bar{F}^{\alpha+\beta}(x,y)\quad dy\, dx\ \right);\quad \alpha > 0\ ,\beta > 0, (16)$$

has the following property: if X and Y are independent, and $\bar{F}(x,y) = \bar{F}(x)\bar{F}(y)$, then, as proposition 3.1, we have

$$\varepsilon H_w^N(X,Y) = \frac{E(y^2)\varepsilon H_w^N(X) + E(x^2)\varepsilon H_w^N(Y)}{2} + \frac{1 - E(y^2)/2 - E(x^2)/2}{(\alpha+\beta)},$$

where $E(x^2) = 2\int_0^\infty x\,\bar{F}(x)\,dx$, and $E(y^2) = 2\int_0^\infty y\,\bar{F}(y)\,dy$.

And then, we introduced the weighted cumulative H^N – entropy with weighted function $w(x) = x$, as:

$$C\varepsilon H_w^N(X) = \frac{1}{\alpha+\beta}\left[1 - \int_0^\infty xF(x)\ln F^{\alpha+\beta}\ (x)dx\right], \alpha > 0. \alpha,\beta \neq 1. \qquad (17)$$

Proposition 3.3. We remember that the two – dimensional version of (17), defined as:

$$C\varepsilon H_w^N(X,Y) = \frac{1}{\alpha+\beta}\left(1 - \int_0^\infty \int_0^\infty xy\,F(x,y)\ \ln F^{\alpha+\beta}(x,y)\quad dy\, dx\ \right);\quad \alpha > 0\ ,\alpha,\beta \neq 1, (18).$$

If X and Y are independent, according to proposition 3.1, we have

$$C\varepsilon H_w^N(X,Y) = \vartheta(y)C\varepsilon H_w^N(X) + \vartheta(X)C\varepsilon H_w^N(Y) + \frac{1-\vartheta(y)-\vartheta(x)}{\alpha+\beta},$$

where $\vartheta(x) = \int_0^\infty x\, F(x)\, dx$, and $\vartheta(y) = \int_0^\infty y\, F(y)\, dy$.

4. STOCHASTIC ORDERS BASED ON H^N- ENTROPY

Stochastic orders have been attracted an increasing number of authors in the last 40 years, they used them in several areas of probability such as reliability theory, survival analysis, operation research, mathematical finance, and risk theory. Then, in this section we explore the probability of application of stochastic orders.

Definition 4.1. Let X and Y be two random variables with density functions $f(x), f(y)$, distribution functions $F(x), F(y)$, survival functions $\bar{F}(x), \bar{F}(y)$, H^N – entropy measures $H^N(X), H^N(Y)$, and weighted H^N – entropy measures $H_w^N(X), H_w^N(Y)$, respectively, if X is less than Y, then,

1. Entropy ordering $(X \leq_e Y)$, if $H^N(X) \leq H^N(Y)$, for all $x \geq 0$.
2. Weighted entropy ordering $(X \leq_{we} Y)$, if $H_w^N(X) \leq H_w^N(Y)$, for all $x \geq 0$.
3. Less uncertainty ordering $(X \leq_{lu} Y)$, if $H^N(X) \leq H^N(Y)$, for all $x \geq 0$.

Definition 4.2. Let X and Y be two random variables**,** if X is less than Y in the usual stochastic order, then,

1. $(X \leq_{st} Y)$, if $E(u(X)) \leq E(u(Y))$, for all non – decreasing function.
2. $(X \leq_{st} Y)$, if $H^N(X) \leq H^N(Y)$, for H^N- non – decreasing entropy function.
3. $(X \leq_{st} Y)$, if $H_w^N(X) \leq H_w^N(Y)$, for H^N- non – decreasing weighted entropy function.
4. If X_i and Y_i are independent sets of random variables with $(X_i \leq_{st} Y_i)$, for each i, then, $H^N(X_1, \ldots\ldots, X_n) \leq H^N(Y_1, \ldots\ldots, Y_2)$.

Definition 4.3. Let X and Y be two random variables**,** if X is less than Y in the convex order, then,

$$E(u(x)) \leq E(u(y)),$$

for any convex function u.

Theorem 4.1. Suppose X_1 , and X_2 are two be two random variables , let $X_1 \leq_e X_2$, then $H^N(X_1) \leq_e H^N(X_2)$ if $X_1 \leq_{cx} X_2$.

Proof : Due to fact that $g(X) = \frac{1}{\alpha+\beta}\left[1 - \ln f^{\alpha+\beta}(x)\right]$, is a convex function, and when $X_1 \leq_{cx} X_2$, we get that $E(g(X_1)) \leq E(g(X_2))$, by apply Eq. (7), we have $H^N(X_1) \leq_e H^N(X_2)$.

Theorem 4.2. If X and Y are non – negative random variables such that $(X \leq_{st} Y)$, then, $C\varepsilon H^N(X) \leq C\varepsilon H^N(Y) \leq C\varepsilon H^N(X, Y)$.

Proof: the proof follows propositions (2.5) and (2.6).

Definition 4.4. Let X and Y are non – negative random variables, with distribution function $F(x), F(y)$, then X is said to be smaller than Y in Laplace transform order $(X \leq_{lt} Y)$, then $H^N(X) \leq H^N(Y)$, where $H^N(X)$, and $H^N(Y)$ are positive functions.

Definition 4.5. Let two absolutely continuous random variables X_1, and X_2 with density function

$$f_i(x) = e^{-a_i x}, a, x > 0.$$

Then:

- If $a_1 \geq a_2$, and $H^N(X) \leq H^N(Y)$, then, $(X \leq_e Y)$.
- If $a_1 \leq a_2$, and $H^N(X) \geq H^N(Y)$, then, $(X \geq_e Y)$.
- If $a_1 \geq a_2$, and $(X \leq_{we} Y)$, then $H_w^N(X) \leq H_w^N(Y)$.
- $H^N(X) \leq H^N(Y)$, when $a_1 \geq a_2$, then $(X \leq_{lt} Y)$, when $H^N(X)$ and $H^N(Y)$ are positive functions.

5. COMPARISON FOR SHANNON ENTROPY AND H^N-ENTROPY MEASURES AND NUMERICAL RESULTS

In this section, we introduce Shannon entropy and H^N- entropy for quasi – lindley, nakagami – μ, chi square, rayleigh, and weighted nakagami – μ distributions in tables 1 and 2, then the relative loss entropy introduced in table 3, after this the numerical results presented in tables 4 – 8.

5.1 H^N – Entropy and Shannon Entropy For Some Distributions

- **H^N – entropy for nakagami – μ distribution**

The probability density function given by:

$$f(x) = \begin{cases} \dfrac{2(\mu/\Omega)^{\mu}\, x^{2\mu-1} e^{\left(-\mu x^2/\Omega\right)}}{\Gamma(\mu)}, x > 0, \\ 0\,, \quad otherwise \end{cases} \tag{19}$$

Then H^N – entropy is given:

$$H^N(X) = \frac{1}{\alpha+\beta}\left(1 - (\alpha+\beta)\frac{2\left(\mu/\Omega\right)^{\mu}}{\Gamma(\mu)}\left[\int_0^\infty x^{2\mu-1} e^{\left(-\mu x^2/\Omega\right)} \ln 2\, dx + \int_0^\infty \mu x^{2\mu-1} e^{\left(-\mu x^2/\Omega\right)} \ln\frac{\mu}{\Omega}\, dx + \int_0^\infty x^{2\mu-1} e^{\left(-\mu x^2/\Omega\right)} (2\mu-1)\ln x\, dx - \int_0^\infty x^{2\mu-1} e^{\left(-\mu x^2/\Omega\right)} \frac{\mu x^2}{\Omega}\, dx - \int_0^\infty x^{2\mu-1} e^{\left(-\mu x^2/\Omega\right)} \ln\Gamma(\mu)\, dx\right]\right),$$

since $\int_0^\infty x^{2\mu-1}\, e^{\left(-\mu x^2/\Omega\right)}\, dx = \Gamma(\mu)/\left[2\left(\frac{\mu}{\Omega}\right)^{\mu}\right]$,

and $\int_0^\infty x^{2\mu-1}\, e^{\left(-\mu x^2/\Omega\right)}\, \ln x\, dx = \frac{1}{4}\,\Gamma(\mu)\left(\frac{\mu}{\Omega}\right)^{-\mu}\left[\psi(\mu) - \ln\frac{\mu}{\Omega}\right]$, then, we have

$$H^N(X) = \left[\frac{1}{(\alpha+\beta)} + \mu\left(1-\psi(\mu)\right) + \frac{1}{2}\psi(\mu) - \frac{1}{2}\,ln\left(\frac{\mu}{\Omega}\right) + ln\left(\frac{\Gamma(\mu)}{2}\right)\right].$$

Then, the H^N – entropy and Shannon entropy for some distributions proposed in table 1 and table 2.

Table 1: H^N – entropy for some different distributions

Distribution	Density function	H^N – entropy
Quasi-lindley	$\frac{\theta(\mu+\theta x)}{\mu+1}e^{-\theta x}$	$\left(\frac{1}{\alpha+\beta} - \frac{\theta}{(\mu+1)}\left[\frac{e^{\mu}\Gamma(2,\mu)}{\theta}\,ln\frac{\theta}{\mu+1} + \frac{e^{\mu}}{\theta}(e^{-\mu}+\Gamma(0,\mu)+\Gamma(2,\mu)\,ln\,\mu) - \frac{e^{\mu}}{\theta}\left(\Gamma(3,\mu)-\mu\Gamma(2,\mu)\right)\right]\right).$
Rayleigh	$2\theta\,x\,e^{-\theta x^2}$	$\frac{1}{\alpha+\beta}\left(1-\left[(\alpha+\beta)\left[ln(2\,\theta)+\frac{1}{2}(\psi(1)-ln\,\theta)-1\right]\right]\right).$
Chi square	$\frac{1}{2^{\frac{p}{2}}\Gamma\left(\frac{p}{2}\right)}x^{\frac{p}{2}-1}e^{\frac{-x}{2}}$	$\frac{1}{\alpha+\beta}\left(1-(\alpha+\beta)\left[ln\frac{1}{2^{\frac{p}{2}}\Gamma\left(\frac{p}{2}\right)} + \left(\frac{p}{2}-1\right)\left[\psi\left(\frac{p}{2}\right)-ln\frac{1}{2}\right]-\frac{p}{2}\right]\right).$
Weighted nakagami-μ	$\frac{2\left(\mu/\Omega\right)^{\mu+\frac{\theta}{2}}x^{2\mu+\theta-1}e^{\left(-\mu x^2/\Omega\right)}}{\Gamma\left(\mu+\theta/2\right)}$	$\left(\frac{1}{\alpha+\beta} - \left[ln\,2 + \left(\mu+\frac{\theta}{2}\right)\left[ln\left(\mu/\Omega\right)-1\right] + \frac{(2\mu+\theta-1)}{2}\left[\psi\left(\mu+\frac{\theta}{2}\right)-ln\left(\mu/\Omega\right)\right] - ln\,\Gamma\left(\mu+\theta/2\right)\right]\right).$

Table 2: Shannon entropy for some different distributions

Distribution	Shannon entropy
Nakagami-μ	$\left[\mu\left(1-\psi(\mu)\right)+\frac{1}{2}\psi(\mu)-\frac{1}{2}\,ln\left(\frac{\mu}{\Omega}\right)+ln\left(\frac{\Gamma(\mu)}{2}\right)\right]$. (see [9]).
Quasi-lindley	$\frac{-e^{\mu}}{(\mu+1)}\left[\Gamma(2,\mu)\,ln\left[\frac{\theta}{(\mu+1)}\right]+\left[e^{-\mu}+\Gamma(0,\mu)+\Gamma(2,\mu)\,ln\,\mu\right]-\left[\Gamma(3,\mu)-\mu\,\Gamma(2,\mu)\right]\right].$
Rayleigh	$\left[1-ln(2\,\theta)-\frac{1}{2}(\psi(1)-ln\,\theta)\right]$. (see [3]).
Chi square	$\left[-ln\frac{1}{2^{\frac{p}{2}}\Gamma\left(\frac{p}{2}\right)}-\left(\frac{p}{2}-1\right)\left[\psi\left(\frac{p}{2}\right)-ln\frac{1}{2}\right]+\frac{p}{2}\right].$
Weighted nakagami-μ	$\left[-ln\,2+\left(\mu+\frac{\theta}{2}\right)\left(1-ln\left(\mu/\Omega\right)\right)-\frac{(2\mu+\theta-1)}{2}\left[\psi\left(\mu+\frac{\theta}{2}\right)-ln\left(\mu/\Omega\right)\right]+ln\,\Gamma\left(\mu+\theta/2\right)\right]$. (see [9]).

5.2 Relative Loss Entropy (RL)

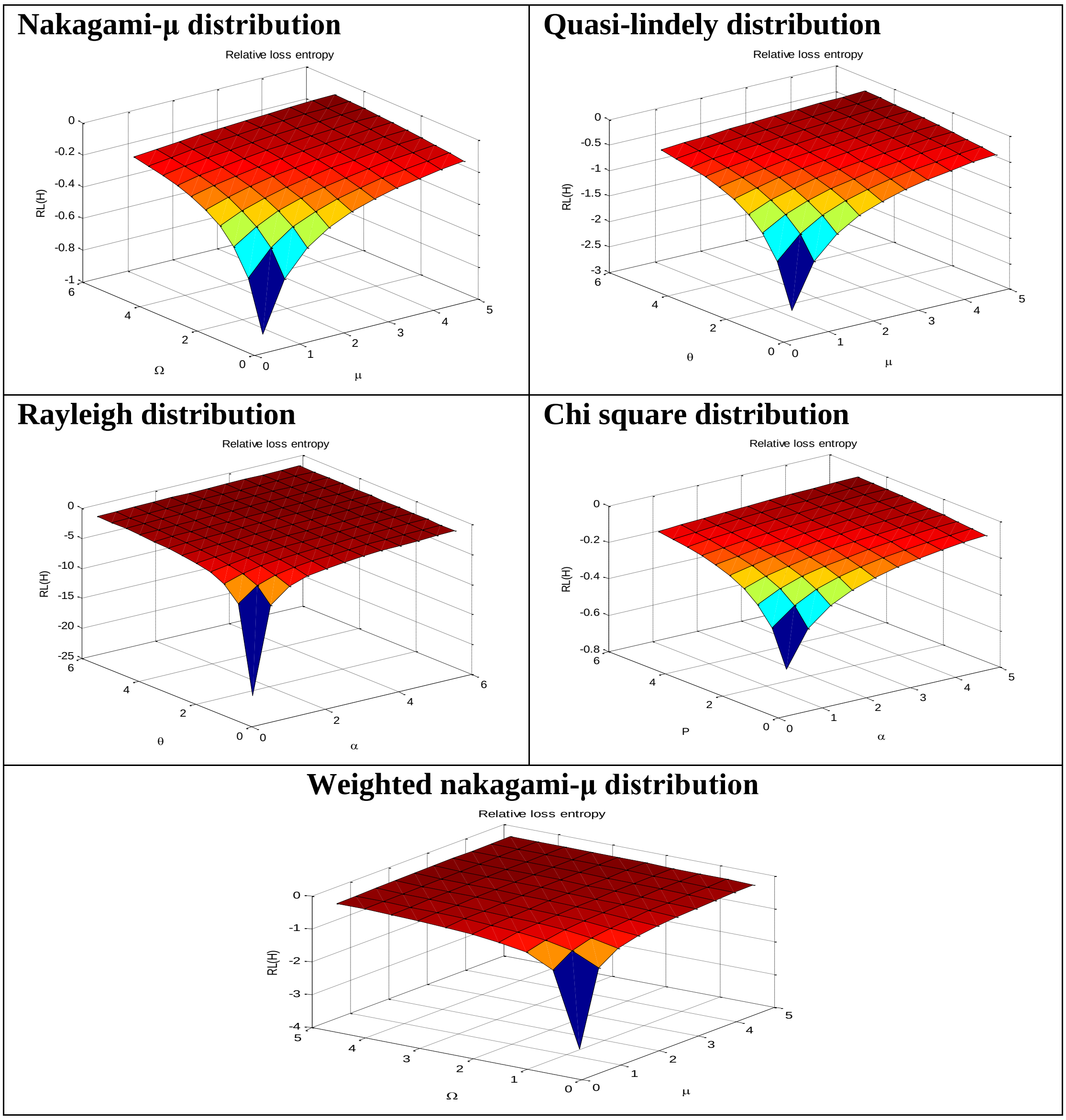
In stochastic process, the probability distribution changes with time, and these changes make the entropy measure also changes. Then, the relative loss entropy measures the amount of changes. In table 3, the relative loss entropy for some distribution, then, the RL is given:

$$RL = \frac{H_{sh}(X)-H^N(X)}{H_{sh}(X)} \tag{20}$$

Table 3: Relative loss entropy for some different distributions

Distribution	RL	IRL	DRL
Nakagami-μ	$RL(N)=1-\frac{\left[\frac{1}{(\alpha+\beta)}+\mu\left(1-\psi(\mu)\right)+\frac{1}{2}\psi(\mu)-\frac{1}{2}\,ln\left(\frac{\mu}{\Omega}\right)+ln\left(\frac{\Gamma(\mu)}{2}\right)\right]}{\left[\mu\left(1-\psi(\mu)\right)+\frac{1}{2}\psi(\mu)-\frac{1}{2}\,ln\left(\frac{\mu}{\Omega}\right)+ln\left(\frac{\Gamma(\mu)}{2}\right)\right]}.$	Ω and β increase	μ increases
Quasi-lindley	$RL(Q)=\frac{H_{sh}(X)-H^{N}(X)}{H_{sh}(X)}$	α and β increase	θ and μ increase
Rayliegh	$RL(R)=1-\frac{\left(\frac{1}{\alpha+\beta}-\left[\left[ln(2\,\theta)+\frac{1}{2}(\psi(1)-ln\,\theta)-1\right]\right]\right)}{\left[1-ln(2\,\theta)-\frac{1}{2}(\psi(1)-ln\,\theta)\right]}.$	α and β increase	θ increases
Chi square	$RL(C)=1-\frac{\left(\frac{1}{\alpha+\beta}-\left[ln\frac{1}{2^{\frac{p}{2}}\Gamma\left(\frac{p}{2}\right)}+\left(\frac{p}{2}-1\right)\left[\psi\left(\frac{p}{2}\right)-ln\frac{1}{2}\right]-\frac{p}{2}\right]\right)}{\left[-\,ln\frac{1}{2^{\frac{p}{2}}\Gamma\left(\frac{p}{2}\right)}-\left(\frac{p}{2}-1\right)\left[\psi\left(\frac{p}{2}\right)-ln\frac{1}{2}\right]+\frac{p}{2}\right]}.$	P, α and β increase	-
Weighted nakagami-μ	$RL(N^{w})=\frac{H_{sh}(X)-H^{N}(X)}{H_{sh}(X)}$	Ω, θ, α and β increase	μ increases

Figure 1: The Relative loss entropy



- The relative loss entropy is always negative, though that, the new entropy is better than shannon.
- With regard to the fixed $\beta = 1$, $RL(Q) > RL(N)$, and $RL(Q) > RL(C)$.
- With regard to the fixed $\Omega = 2$, $RL(N^w) > RL(N)$.
- When $\alpha = 0.5$, and $\beta = 1$,
    - $RL(Q) > RL(N), RL(Q) > RL(C)$.
    - $RL(Q) > RL(R), RL(R) > RL(N)$.
    - $RL(C) > RL(N)$.

CONCLUSIONS

In this paper, new entropy measure of order α and β has been introduced and its properties. This new measure is always positive unlike Shannon entropy, and the H^N-entropy has two parameters, that made it more flexible than Shannon, and the relative loss entropy is always negative, though the new entropy is better than Shannon entropy.

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APPENDICIES

Table 4

The relative loss entropy in using H^N-entropy instead of Shannon entropy for nakagami-μ distribution

μ	Ω	$RL(N)$		
		$\beta = 1$	$\beta = 2$	$\beta = 3$
		$\alpha = 0.5$		
0.5		-0.9185	-0.5511	-0.3937
1		-1.1196	-0.6717	-0.4798
1.5		-1.4919	-0.8952	-0.6394
2		-2.0443	-1.2266	-0.8761
2.5	**1**	-2.9359	-1.7615	-1.2582
3		-4.6382	-2.7829	-1.9878
3.5		-9.2625	-5.5575	-3.9696
4		-73.6382	-44.1829	-31.5592
0.5		-0.6217	-0.3730	-0.2664
1		-0.7077	-0.4246	-0.3033
1.5		-0.8402	-0.5041	-0.3601
2		-0.9911	-0.5946	-0.4247
2.5	**2**	-1.1621	-0.697	-0.4981
3		-1.3597	-0.8158	-0.5827
3.5		-1.5928	-0.9557	-0.6826
4		-1.8746	-1.1248	-0.8034

Table 5

The relative loss entropy in using H^N-entropy instead of Shannon entropy for quasi – lindley distribution

θ	μ	$RL(Q)$		
		$\beta = 1$	$\beta = 2$	$\beta = 3$
		$\alpha = 0.5$		
0.5		-0.2950	-0.1770	-0.1264
1		-0.4256	-0.2554	-0.1824
1.5		-0.5742	-0.3445	-0.2461
2		-0.7634	-0.4581	-0.3272
2.5	**0.1**	-1.0255	-0.6153	-0.4395
3		-1.4251	-0.8551	-0.6108
3.5		-2.1256	-1.2754	-0.9110
4		-3.7015	-2.2209	-1.5864
0.5		-0.2976	-0.1785	-0.1275
1		-0.4309	-0.2585	-0.1847
1.5		-0.5839	-0.3503	-0.2502
2		-0.7805	-0.4683	-0.3345
2.5	**0.2**	-1.0565	-0.6339	-0.4528
3		-1.4858	-0.8915	-0.6368
3.5		-2.2634	-1.3581	-0.9700
4		-4.1406	-2.4844	-1.7745

Table 6

The relative loss entropy in using H^N-entropy instead of Shannon entropy for rayleigh distribution

$\boldsymbol{\theta}$	$\boldsymbol{\alpha}$	$\boldsymbol{RL(R)}$		
		$\beta = 1$	$\beta = 2$	$\beta = 3$
0.1		-0.3817	-0.2290	-0.1636
0.2		-0.4761	-0.2857	-0.2041
0.5		-0.7077	-0.4246	-0.3033
1		-1.1196	-0.6717	-0.4798
1.5	**0.5**	-1.6975	-1.0185	-0.7275
2		-2.6786	-1.6072	-1.1480
2.5		-4.8550	-2.9130	-2.0807
3		-14.4442	-8.6665	-6.1904
0.1		-0.2290	-0.1636	-0.1272
0.2		-0.2857	-0.2041	-0.1587
0.5		-0.4246	-0.3033	-0.2359
1		-0.6717	-0.4798	-0.3732
1.5	**1.5**	-1.0185	-0.7275	-0.5658
2		-1.6072	-1.1480	-0.8929
2.5		-2.9130	-2.0807	-1.6183
3		-8.6665	-6.1904	-4.8147

Table 7

The relative loss entropy in using H^N-entropy instead of Shannon entropy for chi square distribution

P	$\boldsymbol{\alpha}$	$\boldsymbol{RL(C)}$		
		$\beta = 1$	$\beta = 2$	$\beta = 3$
1		-0.8506	-0.5104	-0.3645
1.5		-0.4849	-0.2909	-0.2078
2		-0.3937	-0.2361	-0.1687
2.5		-0.3506	-0.2103	-0.1502
3	**0.5**	-0.3246	-0.1947	-0.1391
3.5		-0.3068	-0.1841	-0.1351
4		-0.2936	-0.1762	-0.1258
4.5		-0.2834	-0.1700	-0.1215
1		-0.5104	-0.3645	-0.2835
1.5		-0.2909	-0.2078	-0.1616
2		-0.2361	-0.1687	-0.1312
2.5		-0.2103	-0.1502	-0.1169
3	**1.5**	-0.1947	-0.1391	-0.1082
3.5		-0.1841	-0.1351	-0.1023
4		-0.1762	-0.1258	-0.0979
4.5		-0.1700	-0.1215	-0.0945

Table 8

The relative loss entropy in using H^N-entropy instead of Shannon entropy for weighted nakagami-μ distribution

θ	μ	Ω	$RL(N^w)$		
			$\beta = 1$	$\beta = 2$	$\beta = 3$
				$\alpha = 0.5$	
0.5			-0.5140	-0.3084	-0.2203
1			-0.4140	-0.2484	-0.1774
1.5			-0.3889	-0.2334	-0.1667
2		**1**	-0.3782	-0.2269	-0.1621
2.5			-0.3725	-0.2235	-0.1596
3			-0.3690	-0.2214	-0.1581
3.5			-0.3666	-0.2199	-0.1571
4			-0.3649	-0.2189	-0.1564
0.5	**0.1**		-0.4056	-0.2434	-0.1738
1			-0.3407	-0.2044	-0.1460
1.5			-0.3235	-0.1941	-0.1387
2		**2**	-0.3161	-0.1897	-0.1355
2.5			-0.3121	-0.1872	-0.1337
3			-0.3096	-0.1857	-0.1327
3.5			-0.3079	-0.1847	-0.1320
4			-0.3067	-0.1840	-0.1314
0.5			-0.5916	-0.3038	-0.2535
1			-0.5064	-0.2876	-0.2170
1.5			-0.4794	-0.2800	-0.2054
2		**1**	-0.4667	-0.2757	-0.2000
2.5			-0.4595	-0.2729	-0.1969
3			-0.4518	-0.2711	-0.1950
3.5			-0.4495	-0.2697	-0.1936
4			-0.3549	-0.2535	-0.1926
0.5	**0.2**		-0.4524	-0.2715	-0.1939
1			-0.4009	-0.2405	-0.1718
1.5			-0.3837	-0.2302	-0.1645
2			-0.3756	-0.2253	-0.1610
2.5		**2**	-0.3709	-0.2225	-0.1590
3			-0.3679	-0.2007	-0.1577
3.5			-0.3658	-0.2195	-0.1568
4			-0.3643	-0.2186	-0.1661