Order Reduction of Large-Scale Dynamical Systems for Benchmark Problems

Rishi Kumar[1], Imran Khan[2]
[1, 2] Department of Electrical Engineering
[1, 2] Azad Institute of Engineering and Technology, Lucknow U.P., India
gitmrishi@gmail.com[1] , pe.imran@gmail.com[2]

Abstract: *This paper aims to introduce a new model order reduction (MOR) method for simplifying the complexity of large-scale stable linear dynamical (LSSL) systems. The Balanced Truncation (BT) and Padé approximation methods are used to obtain the denominator and numerator polynomial coefficients of the reduced-order model (ROM). This method is used to address the shortcomings of the Padé approximation and BT methods. In this procedure, the stability and steady-state value of the LSSL system are guaranteed to be well preserved in the reduced-order model. The proposed technique has been applied successfully to the SISO system and has been extended to multi-dimensional systems. The proposed technique is confirmed by applying it to benchmark examples of 1006th and 120th orders. The results are compared to other well-known methods as well as recent work on performance indices and time domain specifications.*

Keywords: Model Order reduction, Balanced Truncation method, Padé approximation, stability, steady-state values.

1 Introduction

This is especially true in control system design, where the engineer must govern physical systems whose analytical model is expressed as an LSSLS system [1]–[3]. All physical systems, including aircraft, chemical plants, refineries, electrical power systems, urban traffic networks, digital communication networks, and control systems, begin with a mathematical model. Theoretical concerns often lead to a complex and high-order model in practice. As a result of interconnecting multiple interacting subsystems, the resulting system size can be too enormous to handle conveniently. Studies on the dynamic stability of modern linked power systems are a good example. The system equations are linear in dynamic settings, but the number of differential equations describing system performance grows fast with the number of connected machines. Calculating a high-order system requires unique numerical approaches. An examination of such a complex system is time-consuming and costly. The system's complexity makes it difficult to comprehend its behaviour [4]. An uncomfortably high order system may be difficult to analyze, synthesize, or identify. Preliminary design and optimization of such systems are frequently easier if a low-order linear model is developed that approximates the system well. It is therefore desired to replace a high order system with a low order system that retains the original system's time constant, damping ratio, natural frequency, and stability. Thus, model reduction aids system understanding. The model must also be mathematically simple to examine and study the findings. Less computing complexity, less hardware complexity, more practical designs, and simpler control rules are the key goals of reduced order models.

The complexity of a system increases concurrently with the tendency of that system's features to increase. This also increases the structural and dimension complexity of the mathematical models. The model reduction has become an important tool in the design and analysis of LSSL systems [5], [6]. The MOR technique produces a ROM without sacrificing crucial higher-order system control characteristics including stability and steady-state value. The MOR of the LSSL system is one of the most important research subjects in engineering and science. For several decades, lower-order system computation has been a hot topic in control system synthesis and analysis. The study, identification, and synthesis of a mathematical model of an LSSL system can be complex. As a result, MOR can be applied to learn more about LSSL systems. The following are some more reasons for computing reduced order systems:

- It can be utilized to reduce computational efforts in simulation problems.
- For the efficient design of controllers, the reduction of hardware complexity, and the creation of simpler control laws.

1.1 Searching the Literature for Some Existing Methods

Several authors have presented a wide range of model order reduction strategies during the past few decades. Papers [7]–[12]and textbooks [6], [13]–[17] have compiled a comprehensive bibliography on this subject. The model order reduction approaches can be divided into frequency and introduction time-domain methods. There are numerous approaches for obtaining low-order models, but the quality of a model is ultimately determined by how it is used.

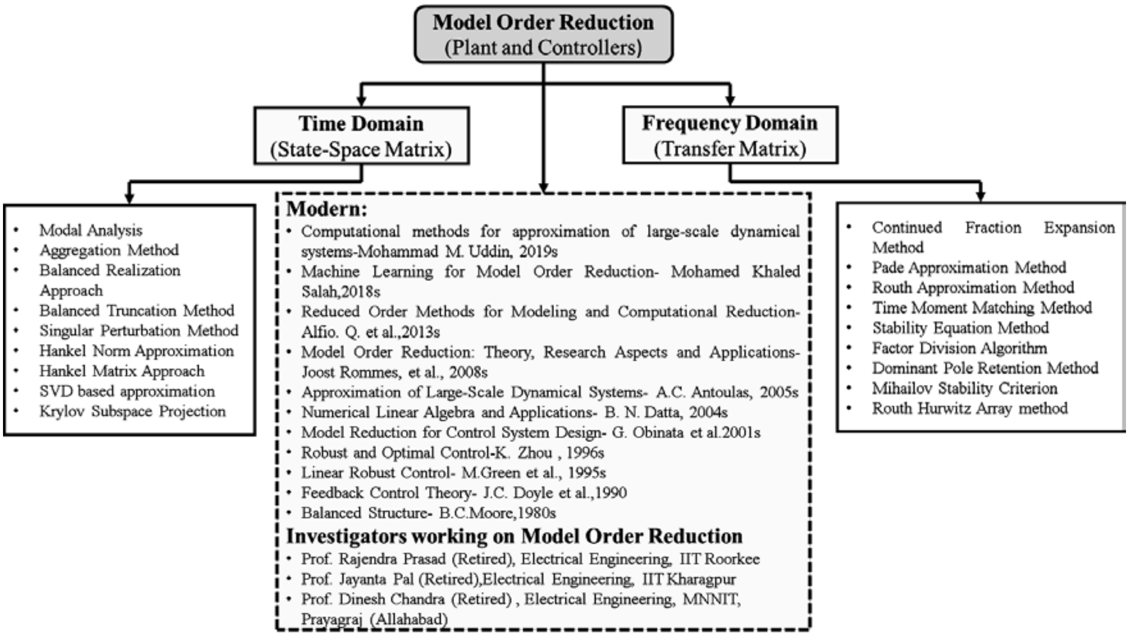

Fig. 1 Evolution of model reduction strategy

The MOR has become a key aspect for analysis and design in current control systems [18]. Currently, the MOR is a widely used technique in engineering design fields such as fluid dynamics, power systems, control theory [1], and so on. The majority of these applications utilize ROM for the development and analysis of large-scale models.

There are numerous strategies available in the literature for both small- and large-scale settings, as well as for frequency and time domain categories. Frequency domain approaches are well suited to system reduction to a moderate order of magnitude, typically less than ten. Because representing the system as a transfer function for more than ten orders will be extremely difficult. Furthermore, the difficulty will be in comprehending, analyzing, and simulating it. The goal of frequency-domain research is to develop new methods rather than to reduce very high-order systems. While time-domain approaches, which characterize the system using a state-space model, are best for reducing very LSSL systems. LSSL systems are frequently referred to as very high order systems.

Many methods are available in the literature, some of which are also mentioned in the preceding paragraph. It has been noted that each method of reduction, whether in the frequency domain or the time domain, has advantages and disadvantages and can be used in a specific situation. In some of the reduction techniques reported in the literature, the instability and mismatch of DC gain are regarded as a major concern. This problem can be solved by using a mixed, composite, or modified algorithm concept in the theory of methods, in which the stability preservation technique calculates the denominator polynomial and the numerator polynomial is obtained using any other reduction technique. Based on the Padé approximation, a reduced-order model for SISO systems was proposed [19]. However, this method suffers from significant drawbacks, such as the reduced model being unstable even with an original stable system. Furthermore, in some situations, the reduced model results in a non-minimal phase system. There are two types of investigations carried out by the investigator in this paper. A new composite/mixed technique for the ROM in the frequency and time domain using the advantages of the balanced truncation method and Padé approximation has been noted and applied in MOR. The method uses the merits of the Balanced Truncation method and the Padé approximation.

Numerous examples have demonstrated a significant improvement in system approximation using the proposed method over conventional methods. The advantage of the methodology is not only that it matches the steady-state value, but it also preserves stability, matches transient states, is efficient in approximation, and is computationally simple. The main characteristics of these algorithms in this paper are that they are faster to convergence, simple to understand, easy to implement and cover the majority of the problem space. This paper presents a new composite method based on Padé approximation in combination with the balanced truncation method. Different Test Systems up to 1006th order of benchmark problems are being considered and many more from the literature to show the improvement in system approximation by present techniques over conventional approaches. Moreover, the reduced-order model so obtained preserves the stability. The results are compared with timely responses and their performance indices compared with the recently reported model order methods.

This paper is organized into five sections. Section 1 includes the method of introduction and the literature study also discusses past work on model order reduction. The issue is described in section 2. Section 3 proposes a modified methodology for reducing the LSSL system, followed by some of the numerical examples and results are discussed and compared with the literature to the validations of the proposed technique. Finally, Section 5 comprises the conclusions and future scope of the work presented.

2 The Problem Formulation

There are two common ways to model and analyze linear systems. The state-space representation of high order dynamic systems is called time-domain representation, whereas the transfer function representation is called frequency domain representation. Frequency domain reduction methods reduce a transfer function or a transfer function matrix, while time-domain reduction methods reduce a high order state-space model. The goal of model order reduction is to find a model that approximates the high order system in some way and responds to similar inputs.

2.1 Time-domain representation in SISO System

Time-domain representation: The state-space description of the system describes high order differential equations by first-order differential equations.

Consider an order linear time-invariant (LTI) system detailed in the time domain and the corresponding ROM in the form of a transfer matrix given by and $G_r(s)$ respectively as given Eq. (1)-Eq. (2) and Eq. (3) -Eq. (4) in this section. The problem is to find ROM in Eq. (3), which approximates the HOS in some sense and preserves the essential features of HOS such that its response matches the response with the HOS as accurately as possible for the same type of inputs. In this work, we have only considered the step input for comparison.

$$\Sigma : \left\{ \begin{array}{l} \frac{dx(t)}{dt} = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{array} \right\} \Leftrightarrow \left(\begin{array}{c:c} A & B \\ C & D \end{array} \right) \tag{1}$$

$$G(s) = C\left[sI_n - A\right]^{-1} B + D$$
(2)

Where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and are constant matrices. Commonly, $x \in \mathbb{R}^n$ denote the state of the system by p inputs and q outputs for a system, matrices of proper size $r << n$. And $G(s)$ is expressed as the transfer function of Eq.(1)[17], [20], [21].The ROM is described in the form of the time domain.

So, the r[th] order ROM by proposed methodology may be obtained from the HOS Eq. (1), which is given by:

$$\Sigma_r : \left\{ \begin{array}{l} \frac{dx_r(t)}{dt} = A_r x_r(t) + B_r u(t) \\ y_r(t) = C_r x_r(t) + D_r u(t) \end{array} \right\} \Leftrightarrow \begin{pmatrix} A_r & B_r \\ C_r & D_r \end{pmatrix} \tag{3}$$

$$G_r(s) = C_r(sI_r - A_r)^{-1}B_r + D_r(1)$$

3 Methodology for Order Reduction

3.1 The BT Method and Padé Approximation Method

The novel composite strategy, which employs the BT method and Padé Approximation to derive a ROM, has been illustrated in below section. 1. It consists of the two phases listed below.

Step 1: Calculate the denominator polynomial coefficient of the ROM using the BT method[21], [22]. Here, we are discussing the MOR method via BT for HOS. In the BT process, the BT method algorithm, as shown in Table 1 below, is referred to as the BT method. [21], [23].

Moore introduced this method of diagonalizing controllability Q_C and observability Q_O matrices via similarity transformations. Q_C and Q_O are symmetric positive definite or semi-definite matrices produced by solving the Lyapunov equations. This method eliminates "weak" subsystems (least controllable and least observable states) that contribute little to the system's impulse response. After then, the model's low-order approximation is applied [24], [25]. The basic idea is that controllability gramians' singular values relate to the amount of energy required to shift the system's proper states. This is done by removing the least controllable and observable states of the system. The HOS has been balanced using a similarity transformation.

Table 1 Balanced Realization Method of Algorithm

Input System: The LSSL system (A, B, C, D)

Output: ROM (A_r, B_r, C_r, D_r)

1. Solve $AQ_C + Q_C A^T = -BB^T \; for \, Q_C$;
2. Solve $A^T Q_O + Q_O A = -CC^T \; for \, Q_O$;
3. Compute Cholesky factors $Q_C = L_C L_C^{\;T}$ and $Q_O = L_O L_O^{\;T}$;
4. Compute SVD: $U \Sigma V^T = L_O^{\;T} L = [U_1, U_2] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} [V_1, V_2]^T$

where $\sum$ is diagonal positive and U, V has orthonormal columns matrix; with $\sum_1 = diag(\xi_1, \cdots\cdots, \xi_i)$, $\sum_2 = diag(\xi_{i+1}, \cdots\cdot, \xi_n)$.

5. Compute the balancing transformation matrices

$X^{-1} = \Sigma^{-\frac{1}{2}} U^T L_O^{\;T}$, $X = L_C V \Sigma^{-\frac{1}{2}}$;

6. From the balancing realization as: $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} = D) = (XAX^{-1}, XB, CX^{-1}, D)$
7. Select number of dominant Hankel singular values **(***HSV***)** will be the order of reduction r and the eigenvalues of the original system and partition $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ conformally;
8. Truncate to form the reduced realization.

Properties

- (A_r, B_r, C_r, D_r) is asymptotically stable
- Error bound: $\|G_r(s) - G(s)\|_{H_\infty} \leq 2(\xi_{r+1} + ... + \xi_n)$

Now the system is balanced using the concept balancing transformation, which is partitioned as [20], [26]–[28]

$$\hat{G}(s) = \left[\begin{array}{c|c} XAX^{-1} & XB \\ \hline CX^{-1} & D \end{array}\right] \quad (9)$$

$$\hat{G}(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] \quad (10)$$

Eq. (11), a $\hat{G}(s)$ has been achieved, the minimal realization of the model partition into a strong and weak subsystem. SPA can therefore be easily applied to subsystems.

$$= \underbrace{\begin{bmatrix} A_{11} & B_1 \\ C_1 & D_r \end{bmatrix}}_{Strong} + \underbrace{\begin{bmatrix} A_{22} & B_2 \\ C_2 & 0 \end{bmatrix}}_{Weak} \Leftrightarrow \Sigma : \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

(11)

where A_{11} and Σ_1 are lower-order matrix, it is part of a strong subsystem which is also ($r<n$). The subsystem (A_{11}, B_1, C_1) must be a good approximation of the balanced system if $\sigma_r \gg \sigma_{r+1}$ proposed by B C. Moore, 1981 [21][29].

$$G_r(s) = C_1(sI_r - A_{11})^{-1} B_1 + D_r \quad (12)$$

It is demonstrated in [21], [30], [31],[37] that the reduced model $G_r(s)$ obtained is always stable. It is also noted that the steady-state value of the ROM can be changed from the steady-state value of the LSSL system.

Steps 2: Calculation of numerator polynomial coefficient of the ROM, using Padé approximation method [27], [32]–[35].

Consider an order Eq. (1) system represented by the following transfer function:

$$G(s) = \frac{N(s)}{D(s)} = \frac{n_0 + n_1 s + n_2 s^2 + \cdots n_{n-1} s^{n-1}}{d_0 + d_1 s + d_2 s^2 + \cdots + d_n s^n} \quad (13)$$

or $$G(s) = \frac{\sum_{i=0}^{n-1} n_i s^i}{\sum_{i=0}^{n} d_i s^i} \quad (14)$$

where n_i, d_i are scalar constants of the LSSL system.

The key aim of this involvement is to calculate the coefficients constraints of r^{th}-order $(r < n)$ reduced system represented as in Eq. (15).

$$G_r(s) = \frac{N_r(s)}{D_r(s)} = \frac{\hat{n}_0 + \hat{n}_1 s + \hat{n}_2 s^2 + \cdots + \hat{n}_{r-1} s^{r-1}}{\hat{d}_0 + \hat{d}_1 s + \hat{d}_2 s^2 + \cdots \hat{d}_{r-1} s^{r-1} + \hat{d}_r s^r} \tag{15}$$

Where are unknown constants of the ROM. These parameters are to be achieved using the proposed method while preserving the significant characteristics of the LSSL system.

Consider Eq. (16) in terms of numerator polynomial and denominator polynomial, which use for the series expansion as follows:

$$G(s) = \sum_{i=0}^{\infty} p_i s^{-i-1} \tag{16}$$

(Expanding about $s = \infty$) : Markov parameters

$$= p_1 s^{-1} + p_2 s^{-2} + p_3 s^{-3} + \cdots + p_n s^{-n} + \cdots \tag{17}$$

$$G(s) = -\sum_{i=0}^{\infty} t_i s^i \tag{18}$$

(Expanding about $s = 0$) : Time Moments

$$= t_1 + t_2 s + t_3 s^2 + \cdots + t_n s^{n-1} + \cdots \tag{19}$$

where p_i and are the i^{th}-Markov and Time moment parameter of the HOS, respectively.

$$G_r(s) = \frac{N_r(s)}{D_r(s)} = \frac{\sum_{j=0}^{r-1} \hat{n}_j s^j}{\sum_{j=0}^{r} \hat{d}_j s^j} \tag{20}$$

$$= \hat{p}_1 s^{-1} + \hat{p}_2 s^{-2} + \hat{p}_3 s^{-3} + \cdots + \hat{p}_n s^{-n} + \cdots \tag{21}$$

$$= \hat{t}_1 + \hat{t}_2 s + \hat{t}_3 s^2 + \cdots + \hat{t}_n s^{n-1} + \cdots \tag{22}$$

The numerator polynomial coefficients are computed by matching the time moment parameters of the Taylor series expansion coefficients about $s = 0$ LSSL system with those of the ROM.

Where t_i $(i = 1, 2, 3...)$ are the Taylor series expansion ($s = 0$).

$$\underbrace{\begin{aligned}
&n_0 = d_0 t_0 \\
&n_1 = d_0 t_1 + d_1 t_0 \\
&n_2 = d_0 t_2 + d_1 t_1 + d_2 t_0 \\
&\vdots \\
&\vdots \\
&n_{\alpha-1} = d_0 t_{\alpha-1} + d_1 t_{\alpha-2} + \cdots + d_{\alpha-2} t_1 + d_{\alpha-1} t_0 \\
&n_{r-\beta} = d_r p_{\beta-1} + d_{r-1} p_{\beta-2} + \cdots + d_{r-\beta+2} p_1 + d_{r-\beta+1} p_0 \\
&n_{r-\beta+1} = d_r p_{\beta-2} + d_{r-1} p_{\beta-3} + \cdots + d_{r-\beta+3} p_1 + d_{r-\beta+2} p_0 \\
&\vdots \\
&n_{r-2} = d_r p_1 + d_{r-1} p_0 \\
&n_{r-1} = d_r p_0
\end{aligned}}_{Pade' \; Sense} \tag{23}$$

The numerator can be calculated by solving the above procedure. Finally, the numerator polynomial coefficient $N_r(s)$ is obtained as

$$N_r(s) = \hat{n}_0 + \hat{n}_1 + \hat{n}_2 s^2 + \cdots \hat{n}_{r-1} s^{r-1} \tag{24}$$

4 Numerical Examples and Results

The efficacy and powerfulness of the suggested method have been illustrated by considering the various SISO/MIMO systems, namely the 1006th order stable example 1, and it has been further extended to the 120th order MIMO system also.

Example 1: Consider the 1006th order FOM model, taken from [36], this model is represented in the form of state-space matrices in Eq. (25).

In of order N=1006, and the system components are given by:

$$A_1 = \begin{pmatrix} -1 & 100 \\ -100 & -1 \end{pmatrix}, A_2 = \begin{pmatrix} -1 & 200 \\ -200 & -1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} -1 & 400 \\ -400 & -1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} -1 & & & \\ & -2 & & \\ & & \cdot & \\ & & & -1000 \end{pmatrix},$$

$$A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & A_3 & \\ & & & A_4 \end{pmatrix}, \quad C = (C_1 \quad C_2),\ C_1 = (10....10) \in \mathbb{R}^6,\ C_2 = (1....1) \in \mathbb{R}^{1000}, \tag{25}$$

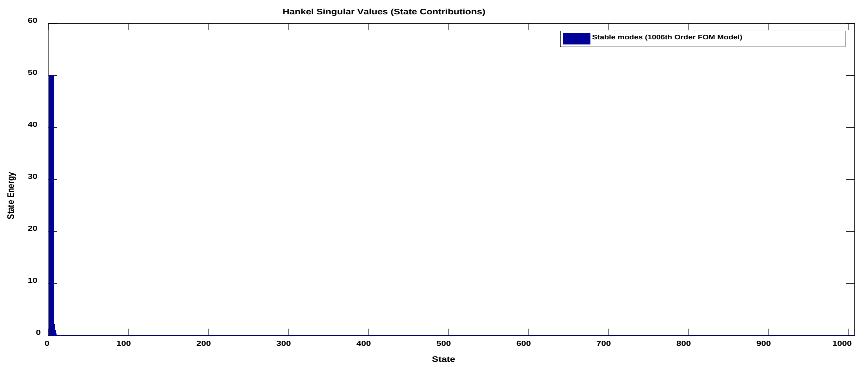

Fig. 2 Bar chart of Hankel Singular Value of 1006th Order Original (LSSL) System

In Fig. 2, the HSV has been calculated and plotted. This bar graph depicts the best reduction order. The original system's order is assumed to be the number of singular values that dominate non-zero. The first eight singular values are critical here, and the third singular values rapidly decay. As a result, the order of reduction has been chosen as an eight order.

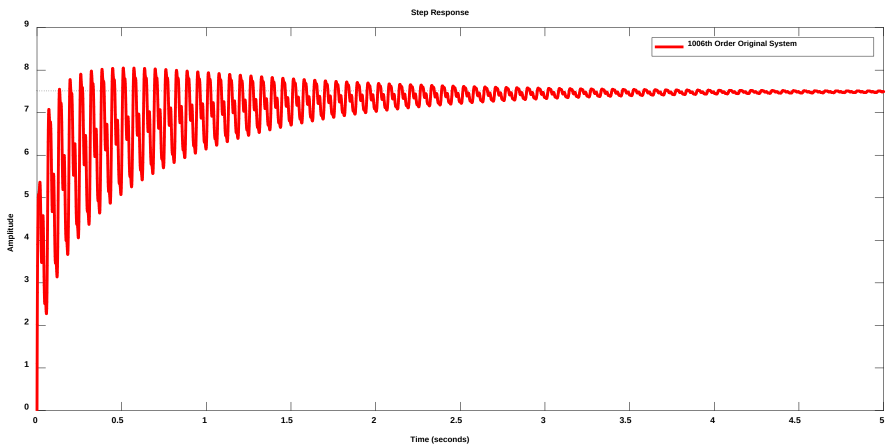

Fig.3 Comparison of Step Responses for Example 1

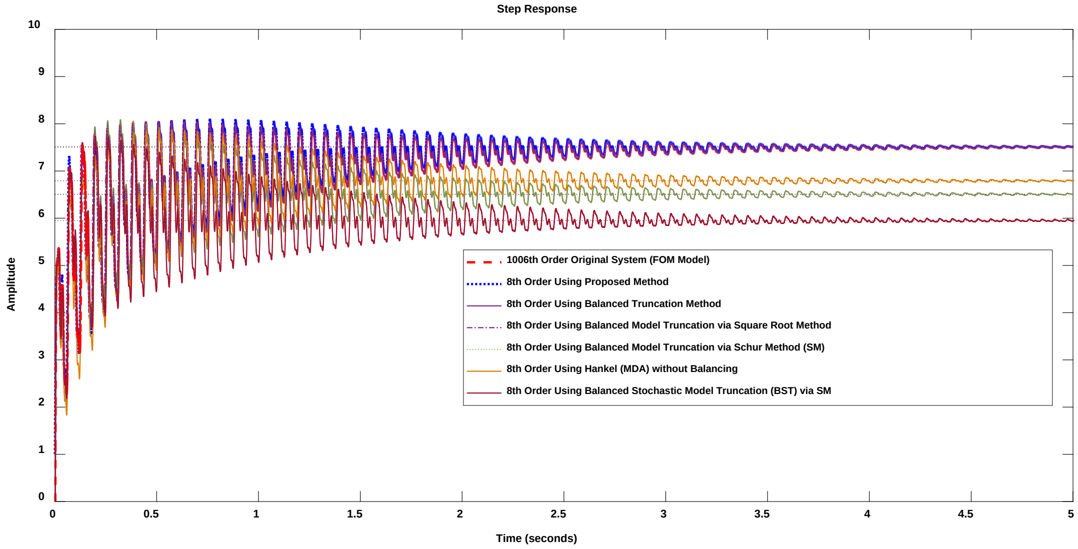

Fig. 4 Comparison of Step Responses for Example 1

Fig.4 depicts the unit step response of the original and reduced-order models, as well as a comparison of the proposed method with another existing method for the 8th order reduced model. It can be seen that the proposed method provides a good approximation when compared to other methods.

4.1 Reduction of MIMO Systems

In this example, to demonstrate the efficacy of the proposed method, a multidimensional benchmark real-world large scale dynamic system is used. The procedure to obtain a reduced order model for a 120th order CD Player large scale MIMO system is the same as for the SISO type system discussed earlier.

Example 2: Consider the 120th order CD Player model, taken from [36], this model is represented in the form of state-space matrices.

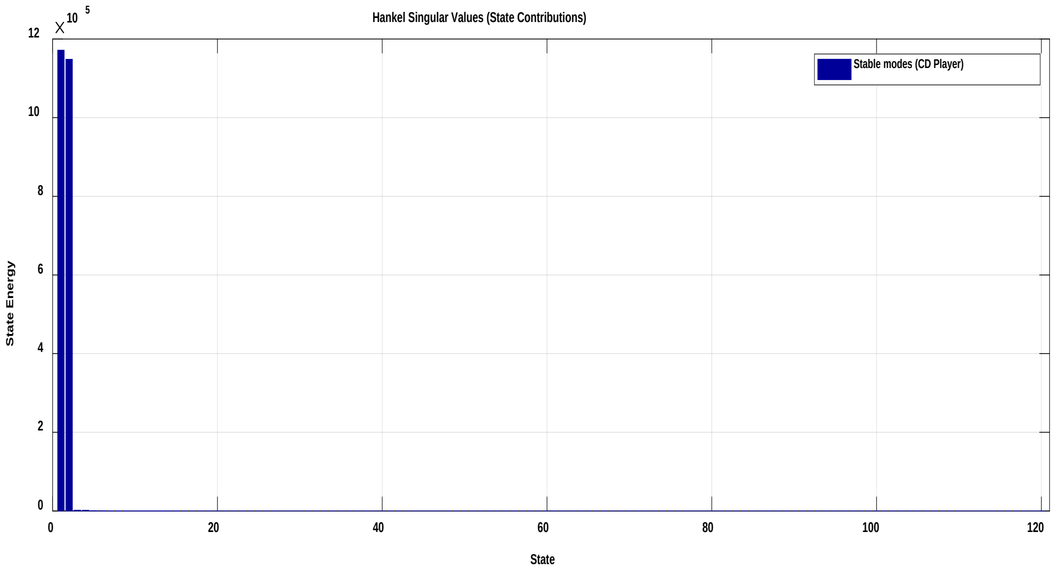

Fig.5 Bar chart of Hankel Singular Value of 120th Order Original System

Fig. 5 shows the HSV calculated and plotted. The optimal reduction order is shown here. The number of non-zero singular values is considered to be an order of the original system. Third singular values gradually fade away. As a result, a six-order reduction is preferred.

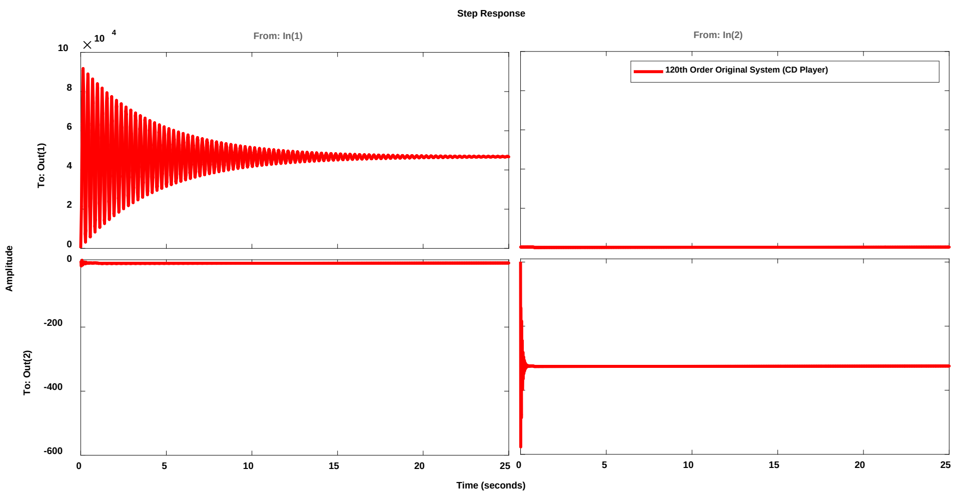

Fig. 6 Original system of Step Responses for example 2

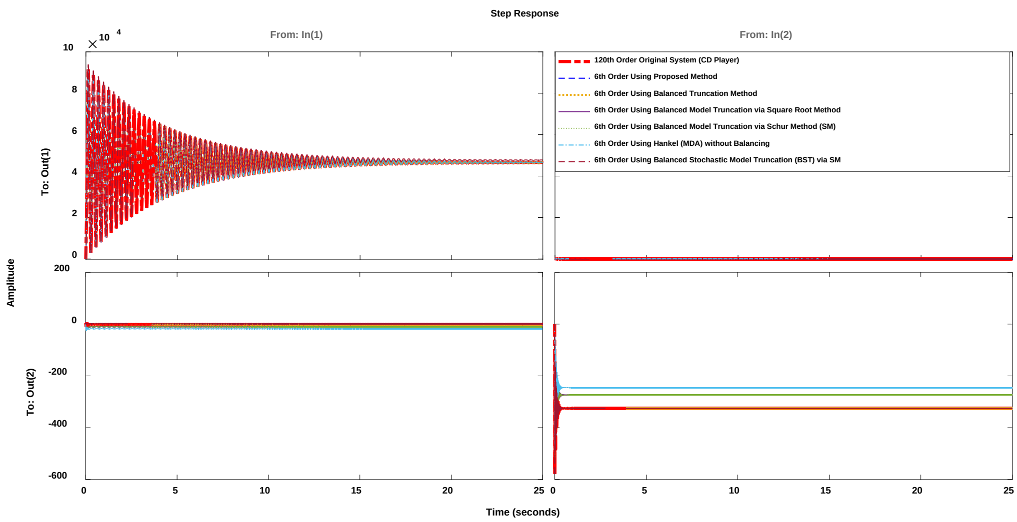

Fig. 7 Comparison of Step Responses for Example 2

Fig. 7 compares the proposed method to alternative methods for 6th order reduced models. As shown in the figure, the reduced model closely approximates the original. The lower order model obtained from the proposed method approximates the original system more closely than the others, as seen in the figures.

5 Conclusion

In this paper, a mixed-method is proposed to overcome the disadvantage of steady-state value mismatch in some systems. This disadvantage was eliminated by combining two methods, such as the balanced truncation method and the pade approximation method. The proposed method has been applied to a large-scale continuous stable SISO system and has also been extended to reduce the benchmark example MIMO system. The Padé approximation is used in combination with the balanced truncation method in these methods. The main premise leads to a better approximation. The method ensures the reduced-order model's stability. Various examples, including benchmark instances, have been used to validate the finding. It should be noted that the ROM obtained using these methods was a close match to the actual system. The findings are compared to previous research on performance indices and step response, as well as other well-known methodologies. Furthermore, when applied to a large-scale system, this method becomes more powerful. As a result of the MOR and controller design research, the following aspects have been identified for future research.

- The method may be investigated to find out the ROM for fractional-order systems.

- It may be possible to examine the simplification of linear interval plants and their controller architecture.
- Preserved the stability and steady-state value of the higher dimensional system in the lower dimensional system.
- It does not require the computation of the initial time moments; it also guarantees to preserve the first "r" time moments of the HOS in its reduced system.
- It is also helpful for the design and processing of controllers and digital signals. The technique proposed here can also simplify the transfer function of LSD discrete-time and interval systems.

Conflicts of interest: The authors declare that they have no conflict of interest.

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