GEO CHROMATIC NUMBER FOR SOME OPERATIONS OF GRAPH

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Abstract

The Geo Chromatic Number (GCN) of $\chi_{gc}(G)$ for some product graphs is derived.

Keywords: Distance, Geodetic, Geodetic number, Union graphs, Intersection graphs, Join Graphs

1. Introduction

The concepts of the GCN of graph were introduced by Beulah Samli and Robinson Chellathurai [1]. In this paper, we consider a graph to be connected, finite, simple where V(G) is node set and E(G) is link set [2,6].

The distance between two nodes x_1, x_2 contained in V(G) is the minimum size of $x_1 - x_2$ paths in G. An $x_1 - x_2$ path of the size $d_G(x_1, x_2)$ is known as geodesic. We indicate $I_G(x_1, x_2)$ as the set of nodes which are lies inside some $x_1 - x_2$ geodesics of G. A node is said to be lie on $x_1 - x_2$ geodetic if c is an inner node of P. The bounded interval $I(x_1, x_2)$ includes x_1, x_2 and all nodes falls on some $x_1 - x_2$ geodesic of G. Consider a non- empty set $S \subseteq V(G)$. For a set I(S) = $\bigcup_{x_1,x_2 \in S} I(x_1, x_2)$. If G is connected graph, thus S is a geodetic set g(S) such that I(S) = V (G) [3]. The minimum cardinality S of G is known geodetic number defined by g(G). A j – node coloring of G is an allotment of j colors to the nodes of G. The coloring is proper if no two joining nodes accept same color such that $\chi(G) = j$ is said to be j – chromatic, where $j \leq k$. a minimum cardinality of a chromatic number of G is known as chromatic set [8, 4, 5].

In this paper, we discussed about Geo Chromatic Number (GCN) [1, 7] for operations on some known graph.

2. Union operation

1.1. Theorem: For any two path graphs p, q > 1 then $\chi_{gc}(G_1 \cup G_2) = \begin{cases} 2, \text{if } m \cup n \text{ is even} \\ 3, \text{if } m \cup n \text{ is odd} \end{cases}$

Proof: Consider the two paths $G' = P_q$ and $G'' = P_p$. Let us label the vertices of P_q by $\{p_1', p_2', p_3', ..., p_q'\}$ and $\{p_1'', p_2'', p_3'', ..., p_p''\}$ be the vertices P_q and the edge set be $\{e_1', e_2', e_3', ..., e_q'\}$ and $\{e_1'', e_2'', e_3'', ..., e_p''\}$. Hence the union of G' and G'' are obtained by taking the union of the vertex and edges. Thus G be the union of G' and G'' are also a path and $\Delta(G) = 2$ and $\delta(G) = 1$ which is two colorable. Thereby we exist following cases:

Case 1: if p = q, here m and n may be odd or even, since the pendant vertices p_1' and p_p' is a geodetic set of G. Here we get two different cases:

Sub case (i): When G = P_{2p}, since S = { p_1', p_p' } is the minimum geodetic set which belongs to different color class, thus S_c together satisfies geodetic set and chromatic set of G. Then $\chi_{gc}(G) = 2$.

Sub Case (ii): When G = P_{2p+1}, S = { p_1', p_p' } is a geodetic set of G. thus S belongs to the same color class, which is not a chromatic set of G. If we choose a vertex belongs to the other color class. Thus its result a chromatic set of G, Thereby S_c form a Geo chromatic set of G. Hence $\chi_{gc}(G) = 3$.

Case (ii): If p > q, thus the result graph is a path graph with p vertices. Let the path Pp may be odd or even. Thus it states the following cases of Case (i).

Case (iii): If p < q, thus the result graph state path graph with q vertices result the case (i).

1.2. Theorem: For any positive integer p, q there exist a vertex x_0 which belongs to both C_p and P_q then

$$\chi_{gc}(C_p \cup P_q) = \begin{cases} 4, if\ p=3, q=2 \\ 3, when\ p = odd, q \geq 2 \\ 3, when\ p \equiv 0\,(\mathrm{mod}\,4), q = odd\ \ and\ \ p \equiv 2\,(\mathrm{mod}\,4), q = even \\ 2\ when\ p \equiv 0\,(\mathrm{mod}\,4), q = even\ and\ p \equiv 2\,(\mathrm{mod}\,4), q = odd \end{cases}.$$

Proof: we take a vertex x_0 which is common for both C_p and P_q . Thereby the vertices of C_p is labeled as $V(C_p)$ = $\{x_0, c_1', c_2', c_3', \ldots\ldots, c_{p-1}'\}$ and the vertices of P_q by $V(P_q)$ = $\{x_0, p_1', p_2', p_3', \ldots\ldots, p_{q-1}'\}$ and the union of C_p and P_q is defined as G' with vertex set $V(C_p \cup P_q)$ = $\{x_0, c_1', c_2', c_3', \ldots\ldots, c_{p-1}', p_1', p_2', p_3', \ldots\ldots, p_{q-1}'\}$. Thus we have p + q – 1 vertex. Let S = $\left\{c_{\frac{p-1}{2}}', c_{\frac{p+1}{2}}', p_{p-1}'\right\}$ or S = $\left\{c_{\frac{p}{2}}', p_{p-1}'\right\}$ be the minimum geodetic set of $C_p \cup P_q$ is G'. The Geo chromatic number can be occur based on the value of p and q as follows:

Case i: When p = 3, q = 2

Let S = $\left\{c_{\frac{p-1}{2}}', c_{\frac{p+1}{2}}', p_{p-1}'\right\}$ be the geodetic set of G' thereby the set of vertices lies in the similar color class and which is not forming a chromatics set. Hence we add a vertex from the various color class to form a geo chromatic set. Thus it states result $\chi_{gc}(G') = 4$.

Case ii: When p = odd, q > 1.

The minimum geodetic set S = $\left\{c_{\frac{p}{2}}', p_{p-1}'\right\}$ which is belongs to the different color classes thus is satisfies the condition of the geo chromatic set and we get $\chi_{gc}(G') = 3$.

Case iii: When $p \equiv 0 \pmod 4$, q = odd and $p \equiv 2 \pmod 4$ q = even.

When q = odd or even, where S = $\left\{ c'_{\frac{p}{2}}, p'_{p-1} \right\}$ which is geodetic set is not a minimum chromatic set, as geodetic set contain the same color class. Hence, to get respective form we can add a vertex which belongs to the different color class, and we get $\chi_{gc}(G') = 3$.

Case iv: When $p \equiv 0 \pmod 4$, q = even and $p \equiv 2 \pmod 4$ q = odd.

When q = odd or even, where S = $\left\{ c'_{\frac{p}{2}}, p'_{p-1} \right\}$ which is geodetic set and also a minimum chromatic set, as geodetic set contain the various color class. Thus it results $\chi_{gc}(G') = 2$.

Corollary 1: Let G = C_m be a cycle with m vertices and H = P_n be a path with n vertices, m >2, n> 1 then the geo chromatic number is Cm and its results $\chi_{gc}(G \cup H) = \begin{cases} 2, if\ m = even \\ 3, if\ m = odd \end{cases}$.

1.3. Theorem: For G = $C_n \cup S_{n-1}$, n>3, then $\chi_{gc}(G) = \begin{cases} \frac{n+1}{2} + 1, n = odd \\ \frac{n}{2} + 2 \quad , n = even \end{cases}$.

Proof: Assume $\{c_1, c_2, c_3, \ldots, c_n\}$ be the vertex set of C_n and v_{n-1} is consider as the center vertex and the extreme vertex of S_{n-1} is named as $\{c_1, c_2, c_3, \ldots, c_{n-2}\}$ and whose diameter is 2. Let us consider the minimum geodetic set S as odd vertex for n is odd and even vertex for n is even. We consider the following cases:

Case i: First assume that n = even, we consider the minimum geodetic set by taking the even vertex of C_n which receives same color classes which is not forming a chromatic set. We add a vertex of different color classes. Thereby it satisfies the condition of geo chromatic set. Hence $\chi_{gc}(G) = \frac{n}{2} + 2$.

Case ii: Suppose we consider n = odd, let us assume the minimum geodetic set by taking the odd vertex of C_n which receives different color classes thus it is not forming a chromatic set. We add a vertex which is belongs to the different color classes. Thus it satisfies the condition. Therefore $\chi_{gc}(G) = \frac{n+1}{2} + 1$.

1.4. Theorem: For the star and cycle graph X = S_p and Y=C_p, p > 4 with V(X) = V(Y) and E(X) = E(Y) be the vertex and edge set of W = X $\cup$ Y then $\chi_{gc}(W) = \begin{cases} 4 \quad , p = 3 \\ \left\lceil \frac{p}{2} \right\rceil + 1, p > 3 \end{cases}$.

Proof: Assume the star graph X = S_p and cycle X = C_p with p vertices and q edges where V(X) = V(Y) = $\{p_1, p_2, p_3, \ldots, p_p\}$ and E(X) = E(Y) = $\{q_1, q_2, q_3, \ldots q_q\}$ such that the union of vertex set V(X) ∪ V(Y) and edge E(X) ∪ E(Y) be the union of X ∪ Y = W. Thereby W becomes a wheel graph of W_n. We consider the following cases:

Case 1: For p = 3, then the union of these graph becomes K_4. For a complete graph K_4 every nodes are consider as geodetic set. Thus S is geodetic and chromatic set of W, Hence its results $\chi_{gc}(W)=4$.

Case 2: For p is even. The set S = {u_i / i ≡1(mod 2)} is the geodetic set of W, and $|S|=\left\lfloor \frac{p}{2} \right\rfloor$. Let us take C be the proper coloring of W such that S = {u_i / i ≡1(mod 2)} was allotted by color say C_1 and S_1 = {u_i / i ≡1(mod 2)} was allotted by color say C_2. The center vertex of W_n was allotted by a new color class say C_3. Now we see that the minimum chromatic set C contains three vertices of distinct colors $|C|=3$. Thus S is not a chromatic set of W. Thereby $|S_c|>\left\lfloor \frac{n}{2} \right\rfloor$, Thus N($u_i$) receive a distinct color class other than S. Let us assume one of the vertex of N(u_i) be u_{i+1}, which is not the center vertex of W. Now S' = {u_i / i ≡1(mod 2)} ∪ {u_{i+1}} is geodetic set of W. Thus $\chi(W)=3$, S' is not a minimum chromatic set of W. If the center vertex of W say $v_0 \in S'$, then the set S_c = {u_i / i ≡1(mod 2)} ∪ {u_{i+1}} ∪ {v_0} is a geo chromatic set of W which satisfies both geodetic set and chromatic set of W, $\chi_{gc}(W)\leq\left\lceil \frac{n}{2} \right\rceil+1$. Hence $\chi_{gc}(W)<\left\lceil \frac{n}{2} \right\rceil+1$ is not true. Therefore it is clear that $\chi_{gc}(W)=\left\lceil \frac{n}{2} \right\rceil+1$.

Case 3: For p is odd, The set S = {u_i / i ≡1(mod 2)} is the geodetic set of W. Assume C be the proper coloring of W such that the center vertex of W_n says v_0 was allotted by a new color which is not allotted in C_{n-1}. It is clear that $|C|=4$. Thus the set S = {u_i / i ≡1(mod 2)} is the geodetic set of W not a chromatic set and such that $\chi_{gc}(W)>\left\lceil \frac{n}{2} \right\rceil$. If $v_0 \in$ S, the set S_c = {u_i / i ≡1(mod 2)} ∪ {v_0} satisfies both geodetic set and chromatic set of W. Hence $\chi_{gc}(W)\leq\left\lceil \frac{n}{2} \right\rceil+1$. Thus $\chi_{gc}(W)<\left\lceil \frac{n}{2} \right\rceil+1$ is not possible. Therefore it results $\chi_{gc}(W)=\left\lceil \frac{n}{2} \right\rceil+1$.

2. Intersection Operation of graph

2.1. Theorem: For any positive integers m, n > 1 then $\chi_{gc}(P_m \cap P_n)=\begin{cases}2, if\ m=n=even\\3, if\ m=n=odd\end{cases}$

Proof : Let us consider G_1' = P_m and G_1'' = P_n , then the intersection of G= $G_1' \cap G_1''$ where as V(G_1') = {$v_1,v_2,v_3.....v_m$}, V(G_1'') = = {$v_1,v_2,v_3.....v_n$} and E(G_1') ={ $e_1,e_2,e_3....e_m$} , E(G_1'') = {$e_1,e_2,e_3......,e_n$} are the two graph then the intersection of these graph are obtained by taking intersection of vertex set and edge set. Thereby, it must have a common edge with a vertex set in G_1' and G_1''. Suppose, if we have a vertex common in both G_1' and G_1''. Then the intersection of G_1' and G_1'' become isolated vertex which is contradict to our result. Therefore we must have one or more common edges and vertices. Thus we have following cases of geo chromatic number

Case 1: when m = n = even
Here we exist a path graph G = P_{2m} (say). Since the set of leaves S = {v_1, v_{2m}} be the minimum geodetic set belongs to the different color classes. Therefore the set S_c = {v_1, v_{2m}} is Geo chromatic set of G. then $\chi_{gc}(G) = 2$.

Case 2: when $m \neq n$ both m and n, either odd or even. We have the following sub cases:

Sub case i: when m < n, if m = odd, n = even. Here there exist a path graph G = min {m,n} = m is odd. Since G is a path thus the geodetic set S = {v_1, v_m} which is minimum belongs to the same color class. Let us choose a vertex belongs to the different color classes v_i (say). Thus S_c = {v_1, v_m} ∪ {v_i} is geo chromatic set of G. It results $\chi_{gc}(G) = 3$.

Sub case ii: when m < n, if m = even, n = odd. Here there exist a path graph G = min {m,n} = m is even. Since G is an even path the geodetic set S = {v_1, v_m} which is minimum and also belongs to the different color class. Hence it results $\chi_{gc}(G) = 2$.

Sub case iii: when m > n, if m = odd, n = even there exist a path graph which is even. Thus its result sub case (ii)

Sub case iv: when m>n, if m = even, n = odd there exist a path graph which is odd. Thus its result sub cases (i).

Case 3: when m = n = odd
Since m and n are odd , G is bipartite and hence 2 – colorable also S = {v_1, v_m} is a minimum geodetic set of G, but which is not a chromatic set, that is v_1 and v_m belongs to the same color classes. We see that one color class is missing in S. by adding the different color to S. S_c satisfies the condition. Hence it results $\chi_{gc}(G) = 3$.

Corollary 2: If G = C_n be a cycle with n vertices and H = P_m be a path with m vertices their obtained a path graph by taking $G' = C_n \cap P_m$. Then the geo chromatic number of the graph is obtained by $\chi_{gc}(G') = \begin{cases} 2, if\ n = even \\ 3, if\ n = odd \end{cases}$.

2.2. Theorem: Let G = W_n be the wheel graph and H = S_n be the star graph and the graph with vertex set V = $V_1 \cap V_2$ and the edge set E = E1 ∩ E2 be the intersection of G' = G ∩ H. then the geo chromatic number of $\chi_{gc}(G') = n$, where n > 3.

Proof: Let G = W_n be the wheel graph and H = S_n be the star graph with n vertex of the graph G' with the vertex set V = $V_1 \cap V_2$ and the edge set E = $E_1 \cap E_2$ be the intersection of the graph G' = G ∩ H. Thus G' is the star graph S_n. Let G' = S_n with n > 3. Hence the set of all pendant vertices is the minimum geodetic set S for G' but S is not a chromatic set of G' and so $\chi_{gc}(G') \geq n$. If the neighborhood of all pendant vertices of G' is contained in S, then S is a geodetic set and also chromatic set of G'. The set S_c = {v_1, v_2, v_3,….,v_{n-1}} u { v_0} is the geo chromatic set of G'. Hence it is clear that $\chi_{gc}(G') = n$.

Remark 1: if C_m and C_n where m = n and all the vertices and edges of C_n and C_m are common then the union and intersection of the cycle graph represents a cycle. Then the result of the geo chromatic number is proved in theorem 5.

Remark 2: if C_n and C_m be the two cycles with n and m vertices their exist some common vertex and edge set. Therefore the intersections of two paths graph again a path and the result of geo chromatic number of the path is resulted in theorem 6.

3. Join graphs:

3.1. Theorem: for G = $P_m + P_n$, m =2 and n>1, then $\chi_{gc}(G) = \begin{cases} 4 & , m = n = 2 \\ \frac{m+n}{2} + 2 & , m = 2, n = even \\ \frac{m+n+1}{2} + 2 & , m = 2, n = odd \end{cases}$.

Proof: Let us consider $V(P_m)$ = { a, b} for path pm where m =2 and $V(P_n)$ = { 1,2,3,….n} be the vertices of P_n where n > 1. The vertices of a join graph of $P_m + P_n$ is {a, b, 1, 2, 3, … , n} and it is 4 colorable and whose diameter is 2. Here we consider the geodetic set S as an odd vertex for when m is odd and when m is even we take m with S which is minimum geodetic set. Thus, we geo chromatic number based on the n>1.

Case 1: when m = n = 2

When we find the join graph between $P_2 + P_2$ we obtain the complete graph. Every vertex in the complete graph is a minimum geodetic set S, thus g(S) = 4 and chromatic number of $P_2 + P_2$ is 4. Therefore it's clear that $\chi_{gc}(G) = 4$.

Case ii: For n = even

Let us find the join graph of $P_2 + P_n$ when n is even the chromatic number of the following graph is 4. Let us consider the minimum geodetic set S by taking the odd vertex of P_n and add the nth vertex along with S where as S receives the different color class but it does not receive the color class of P_2 thereby which is not a minimum chromatic set. Hence by adding the both the color classes of P_2 with S. Therefore S_c satisfies the geo chromatic set. Thus it is results $\chi_{gc}(G) = \frac{m+n}{2} + 2$.

Case iii: For n = odd.

We consider the minimum geodetic set by taking the odd vertex of P_n which receives same colors. Thus it does not satisfy the condition. Hence we add the different colors of P_n with S thus it not satisfies the condition. Thereby we add the remaining color class of P_2 with S which satisfies the condition of chromatic set. Hence S_c is both geodetic and chromatic set of G. thus it states $\chi_{gc}(G) = \frac{m+n+1}{2} + 2$.

3.2. Theorem: For G = $C_n + P_m$, m =2 and n>3, then $\chi_{gc}(G) = \begin{cases} \frac{m+n}{2} + 2 & , m = 2, n = even \\ \frac{m+n-1}{2} + 2 & , m = 2, n = odd \end{cases}$.

Proof:

Assume G = $C_n + P_m$, where m =2 and n > 4. Let {a, b} be the vertex set of P_2 and {1,2,3,4….n}. The vertices of a join graph of $C_n + P_2$ is {a, b, 1, 2, 3, … , n} and whose diameter is 2. We consider the minimum geodetic set by taking the odd vertex of Cn.

Case i: First assume that m =2 and n = even, we consider the minimum geodetic set by taking the odd vertex of C_n which receives same color classes which is not forming a chromatic set. we add a vertex of different color classes from the C_n and also vertices from path P_2. Thereby it satisfies the condition of geo chromatic set. Hence $\chi_{gc}(G) = \frac{m+n}{2} + 2$.

Case ii: Suppose we consider m =2 and n = odd, let us assume the minimum geodetic set by taking the odd vertex of C_n which receives different color classes thus it is not forming a chromatic set. We add the vertices of P_2 which is belongs to the different color classes to S. thus it satisfies the condition. Therefore $\chi_{gc}(G) = \frac{m+n-1}{2} + 2$.

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