Δ-Optimum Exclusive sum labeling of Book graph (B_n)

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Abstract

A graph $G = (V, E)$ is called sum graph if there exists an injective function L called sum labeling, from V to a set of positive integers such that for $x, y \in V$, $xy \in E$ if and only if $L(x) + L(y) = L(w)$ for some vertex w in V. In such case w is called *working vertex* of graph G. A sum labeling of a graph $G \cup \overline{K_r}$ for some positive integer r is said to be *exclusive* with respect to G if all its working vertices are in $\overline{K_r}$. In order to get lebeled exclusively; every graph G needs some isolated vertices. The smallest number of such isolates that need to be added to a graph G is called the *exclusive sum number* of the graph G; denoted by $\epsilon(G)$. The number of isolates must be at least $\Delta(G)$, the maximum vertex degree in G. In case $\epsilon(G) = \Delta(G)$, then G is said to be a Δ-*optimum exclusive sum graph* and the exclusive sum labeling of G using $\Delta(G)$ isolates is called Δ-*optimum exclusive sum labeling* of G.
In this paper we find exclusive sum number of Book graph, B_n and show that it is a Δ-optimum exclusive sum graph. We also define a Δ-optimum exclusive sum labeling for Book graph.

Key words :Book graph, Exclusive sum labeling, Exclusive sum number, Δ-optimum exclusive sum number, Δ-optimum exclusive sum labeling.

1 Introduction

The concept of sum labeling was introduced by Harary in 1990 [4]. A *sum graph* is a simple undirected graph G with a labeling L of its vertices by distinct positive integers such that any two distinct vertices u, v are

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adjacent if and only if $L(u) + L(v) = L(w)$, where w is a vertex in G [4]. In such case w is called a working vertex and L is called a sum labeling of graph G.

The vertex that is not adjacent to any other vertex is called *isolate*. As in every sum graph there needs to be at least one isolated vertex, particularly the vertex with the largest label so every sum graph is disconnected. If any graph G is not sum graph, adding finite number of isolated vertices to it we can always yield a sum graph. The *sum number* $\sigma(G)$ of a connected graph G is the minimum number of isolated vertices that are required to be added to G to yield a sum graph. A graph G together with its minimum number of isolates $\overline{K_{\sigma(G)}}$, is called *optimal summable graph* and corresponding sum labeling of $G \cup \overline{K_{\sigma(G)}}$ is called an *optimal sum labeling* of G [13].

If L is a sum labeling of $G \cup \overline{K_r}$ in such a way that G contains no working vertex then L is said to be an exclusive sum labeling of G. A Sum labeling of a graph $G \cup \overline{K_r}$ for some positive integer r is said to be *exclusive* with respect to G if all of its working vertices are in $\overline{K_r}$. Every connected graph G will require some isolated vertices called isolates to be added so that the graph G together with the additional isolates can support an exclusive sum labeling of a graph G. The least number of such isolates that need to be added to a graph G is called the exclusive sum number of graph G and it is denoted by $\epsilon(G)$. [3],[12]. Note that $\epsilon(G) \geq \sigma(G)$, that is the exclusive sum number is never smaller than the corresponding sum number of graph, as every exclusive sum graph is a sum graph.

Let $\Delta(G)$ be the maximum degree of the vertices of a graph G. Then $\epsilon(G) \geq \Delta(G)$, that is $\Delta(G)$ is a lower bound for $\epsilon(G)$. In case $\epsilon(G) = \Delta(G)$, the graph G is said to be a Δ-optimum exclusive sum graph [7],[10]. For several classes of graph whose exclusive sum number is known, we refer to Gallian survey [3]. The notion of exclusive sum labeling was introduced by Bergstrand et al.[2]. Miller et al.[7] extended the idea to include all graphs. Bergstrand et al.[2] have proved that $\epsilon(K_n) = 2n - 3$, for $n \geq 3$. M.Miller et al.[7], [8] have proven that $\epsilon(P_n) = 2$, for $n \geq 3; \epsilon(C_n) = 3$, for $n \geq 3$. Hartsfield and Smyth in 1995 [6] gave the lower bound for exclusive sum number of complete bipertite graph as $\epsilon(K_{m,n}) \geq m + n - 1$, for $m \geq 2; n \geq 2$ then later it was proved by Miller that $\epsilon(K_{m,n}) = m + n - 1$, for $m \geq 2; n \geq 2$ [7]. Miller et al. have given construction of exclusive sum labeling for various classes of graphs and have found their exclusive sum numbers, such as $\epsilon(H_{2,n}) = 4n - 5$ (cocktail party graph) [9], $\epsilon(F_n) = n$ (fan of order $n + 1$), for $n \geq 4; \epsilon(W_n) = n, \epsilon(C_3^{(n)}) = 2n$ (friendship graph), for $n \geq 2; S_n = n$ (star of order $n + 1$), $\epsilon(S_{m,n}) = max\{m, n\}$ (double star)[3], ϵ(caterpillar G) $= \Delta(G)$ and ϵ (Shrub G) $= \Delta(G)$[13]. The authors have defined exclusive sum labeling for Prism graph (D_n) and proven that $\epsilon(D_n) = 5$ [11].

We in this paper show that Book graph, B_n is a Δ-optimum exclusive sum graph. We also define a Δ-optimum exclusive sum labeling for Book graph. A *Book graph* B_n (see fig. 1) is the Cartesian product $S_n \times P_2$ where S_n is the star with n edges and P_2 is the path graph on two nodes [3]. The motivation for this paper is the survey on some open problems on graph labelings by S. Arumugam et.al [1]. For all the terms and terminologies used here we refer to Harary [5].

2 Exclusive sum number of Book graph (B_n)

In this section we will give an exclusive sum labeling of Book graph B_n with exactly $\Delta(B_n)$ number of isolates, so we can conclude that Book graph (B_n) is Δ-optimum exclusive sum graph.

Let B_n be a book with n pages. There is only one edge say cc' which is common edge for all pages. Then $'n+1'$ is the degree of both the vertices c and c'. Hence $\Delta(B_n) = n+1$. Let v_i and v_i' be the remaining two vertices of i^{th} page which are adjacent to two vertices c and c' respectively. Note here that we are identifing vertices with their labels under L and for the sake of simplicity, simply writing v instead of $L(v)$.

As Book graph is connected, in order to achieve exclusive sum labeling of graph we need to add isolates in the graph. The vertex c is adjacent to $n+1$ vertices, n vertices from v_1 to v_n and $(n+1)^{th}$ is c'. As all these are distinct, we must have at least $n+1$ different sums namely $c+v_1,\ c+v_2, ...,\ c+v_n,\ c+c'$. Thus minimum $\Delta(B_n) = n+1$ isolates are required for exclusive sum labeling of book graph B_n.

Next we will prove that it is exact $\Delta(B_n)$. For that we need to give an exclusive sum labeling of book graph with exactly Δ number of isolates and we are done.

We proceed for the proof in the way that first we provide a labeling and then in order to fulfill the 'if part' of the definition, we define an isolate w for every two adjacent vertices u and v such that $L(u) + L(v) = L(w)$ and show that there are exactly Δ number of such isolates. Secondly we verify the 'only if' part, that the equation $L(u) + L(v) = L(w)$ is satisfied only when u and v are adjacent. As declared earlier we will be writing just u instead $L(u)$.

It is necessary to deal with the cases when n is odd and n is even seperately. We call book B_n as odd book when n is odd and even book when n is even.

2.1 Odd Books

Theorem 1. *$\epsilon(B_n) = \Delta(B_n)$ for all $n \geq 2, n$ odd.*

We label the vertices of B_n where n is odd as follows,

$$\begin{aligned} v_i &= 32i - 31 \quad \text{for } 1 \leq i \leq n-1 \text{ and} \\ v_n &= 16n - 39, \\ v'_i &= 32(n-i) - 19 \quad \text{for } 1 \leq i \leq n-1 \text{ and} \\ v'_n &= 16n - 27, \\ c &= 16n - 19 \text{ and } c' = 16n - 31. \end{aligned}$$

Next to show that this labeling requires exactly $n+1$ isolates, we sum each pair of adjacent vertices of B_n, by considering different cases as follows.

1.

$$\begin{aligned} c + v_i &= 16n - 19 + 32i - 31 \quad \text{for } 1 \leq i \leq n-1 \\ &= \mathbf{16n + 32i - 50} \quad \text{for } 1 \leq i \leq n-1. \\ c + v_n &= 16n - 19 + 16n - 39 \\ &= \mathbf{32n - 58}. \end{aligned}$$

2.

$$\begin{aligned} c' + v'_i &= 16n - 31 + 32(n-i) - 19 \quad \text{for } 1 \leq i \leq n-1 \\ &= 16n + 32(n-i) - 50 \quad \text{for } 1 \leq n-i \leq n-1. \\ &= 16n + 32k - 50 \quad \text{for } 1 \leq k \leq n-1. \\ c' + v'_n &= 16n - 31 + 16n - 27 \\ &= 32n - 58. \end{aligned}$$

3.

$$\begin{aligned} c + c' &= 16n - 19 + 16n - 31, \\ &= \mathbf{32n - 50}. \\ v_i + v'_i &= 32i - 31 + 32(n-i) - 19 \quad \text{for } 1 \leq i \leq n-1. \\ &= 32n - 50 \\ v_n + v'_n &= 16n - 39 + 16n - 27 \\ &= 32n - 66. \end{aligned}$$

Note here that the sum $32n - 66$ is equal to $16n + 32i - 50$ for $i = (n-1)/2$.

Thus we can see that in case of odd book there are exactly $n+1$ distinct vertex sums of adjacent vertices, namely $w_i = 16n + 32i - 50$ for $1 \leq i \leq$

$n-1$, $w_n = 32n-58$ and $w_{n+1} = 32n-50$. They all have to be isolates as all the labels of vertices of Book are odd so these vertex sums are even. Thus the construction of an exclusive labeling for odd Book requires exactly $\Delta = n+1$ isolates. Now adding the above isolates we claim that the graph $G = B_n \cup \{w_1, w_2, ..., w_n+1\}$ where n odd is an exclusive sum graph with the labeling defined above.

Note that the way graph G is constructed, the first part of the definition of exclusive sum graph that, if there is an edge between two vertices u and v of the graph G then there is an isolate in the graph with label $u+v$ is already satisfied. Now for the converse we must check that, if there is no edge between two vertices u and v of G then no vertex of the graph G should have label equal to the sum $u+v$. That means we need to check that, no vertex in the graph $G = B_n \cup \{w_1, w_2, ..., w_n+1\}$ has its label equal to sum of any two non adjacent vertex labels in the graph. In order to prove this we make use of following observations about the labeling:

Observation 1: All the labels of vertices of Book are odd.

Observation 2: All isolate labels are even.

Observation 3: $v_i \equiv 1 \pmod 4, \forall\ i$.
$v'_i \equiv 1 \pmod 4, \forall\ i$
$c, c' \equiv 1 \pmod 4, \forall\ i$ and
$w_i \equiv 2 \pmod 4, \forall\ i$.

It is easy to see that no vertex label of B_n can be the sum of any two vertex labels on B_n (by Observation 1), or the sum of any two isolates (by Observation 2). Also due to Observation 3, it can not be equal to sum of an isolate and a vertex label on B_n as the sum is congruent $3 \pmod 4$. In short no vertex label of B_n is equal to sum of any two vertex labels of the graph G. Also concluding that no vertex of B_n is working vertex.

Now it remains to check that no isolate label is sum of non adjacent vertex labels in the graph G. For that we consider different cases of non-adjacent vertices as follows:

Case 1. Two non adjacent vertices on B_n.

Pair of non adjacent vertices of B_n gives 11 different cases which are characterised into following subcases depending on their sums.

Subcase I. c and vertex non adjacent to it,

1. $c + v'_i = 16n + 32(n-i) - 38$, for $1 \le i < n$.
2. $c + v'_n = 32n - 46$.

Subcase II. c' and vertex non adjacent to it,

3. $c' + v_i = 16n + 32i - 62$, for $1 \leq i < n$.
4. $c' + v_n = 32n - 70$.

Subcase III. non adjacent pairs of v_i s

5. $v_i + v_j = 32(i + j) - 62$, where $1 \leq i \neq j < n$.
6. $v_i + v_n = 16n + 32i - 70$, for $1 \leq i < n$.

Subcase IV. non adjacent pairs of v'_i s

7. $v'_i + v'_j = 32(2n - i - j) - 38$, where $1 \leq i \neq j < n$.
8. $v'_i + v'_n = 48n - 32i - 46$, for $1 \leq i < n$.

Subcase IV. Lastly

9. $v_i + v'_j = 32(n + i - j) - 50$, where $1 \leq i \neq j < n$.
10. $v_i + v'_n = 16n + 32i - 58$, for $1 \leq i < n$.
11. $v_n + v'_j = 48n - 32j - 58$, for $1 \leq j < n$.

It can be easily observed that none of these sums can be equal to the isolate labels.

Case 2. Any two isolates,
the sum is congruent to 0 (mod 4), so can not be equal to any isolate label (by Observation 3).

Case 3. Sum of a vertex label of B_n and an isolate
is congruent to 3 (mod 4), so can not be equal to any isolate label (by Observation 3).

Thus we conclude that no isolate in G is sum of two non adjacent vertex labels in the graph G. This ends the verification for converse part of definition of exclusive sum graph.
So finally we conclude that with the above labeling the graph $G = B_n \cup \{w_1, ...w_{n-1}, w_n, w_{n+1}\}$, for odd n is an exclusive sum graph and the above labeling is a Δ-optimum exclusive sum labeling of Book B_n for n odd with $\epsilon(B_n) = n + 1 = \Delta$. Thus proving that the book graph B_n for n odd, is Δ-optimum exclusive sum graph.
Figure 1 shows example of exclusive sum labeling for Book B_5.

2.2 Even Books

Theorem 2. *$\epsilon(B_n) = \Delta(B_n)$ for all $n \geq 2, n$ even.*

We label the vertices of B_n where n is even as follows,

$$\begin{aligned} v_i &= 16i - 15 \quad \text{for} \;\; 1 \leq i \leq n, \\ v'_i &= 16(n - i) + 5 \;\; \text{for} \;\; 1 \leq i \leq n, \\ c &= 8n - 3 \;\; \text{and} \;\; c' = 8n - 7. \end{aligned}$$

Next to show that this labeling requires exactly $n+1$ isolates, we sum each pair of adjacent vertices of B_n, by considering different cases as follows.

1.

$$\begin{aligned} c + v_i &= 8n - 3 + 16i - 15 \quad \text{for} \quad 1 \leq i \leq n \\ &= \mathbf{8n + 16i - 18} \quad \text{for} \quad 1 \leq i \leq n. \end{aligned}$$

2.

$$\begin{aligned} c' + v_i' &= 8n - 7 + 16(n-i) + 5 \quad \text{for} \quad 1 \leq i \leq n \\ &= 8n + 16(n-i+1) - 18 \quad \text{for} \quad 1 \leq n-i+1 \leq n. \\ &= 8n + 16k - 18 \quad \text{for} \quad 1 \leq k \leq n. \end{aligned}$$

3.

$$\begin{aligned} c + c' &= 8n - 3 + 8n - 7, \\ &= \mathbf{16n - 10}. \\ v_i + v_i' &= 16i - 15 + 16(n-i) + 5 \quad \text{for} \quad 1 \leq i \leq n. \\ &= 16n - 10 \end{aligned}$$

Thus we can see that in case of even book there are exactly $n+1$ distinct vertex sums of adjacent vertices, namely $w_i = 8n + 16i - 18$ for $1 \leq i \leq n$, and $w_{n+1} = 16n - 10$.
Similar to odd case it can be seen that book B_n for n even with above Δ number of isolates yield an exclusive sum graph. Thus the above labeling is Δ-optimum exclusive sum labeling of even book. So we can say that book graph B_n for n even is also a Δ-optimum exclusive sum graph. Thus concluding Theorem 2. Figure 2 shows example of exclusive sum labeling for book graph B_6.
We get the following result.

Theorem 3. *The book graph B_n , for $n \geq 2$ is Δ-optimum exclusive sum graph.*

2.3 Illustration

In figure 1 and 2 we give Δ-optimum exclusive sum labeling for book B_5 and B_6.

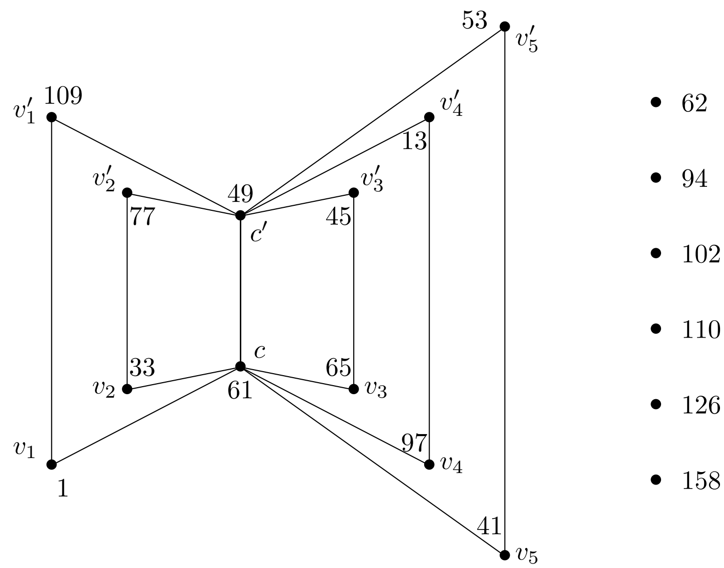

Figure 1: A Δ-optimum exclusive sum labeling for Book graph B_5

References

[1] S. Arumugam, Martin Baca, Dalibor Froncek, Joe Ryan,Some open problems on graph labelings, AKCE Int. J. Graphs Comb., 10, No. 2 (2013), pp. 237-243.

[2] D. Bergstrand, F. Harary, K. Hodges, G. Jennings, L. Kuklinski, J. Wiener, The sum number of a complete graph, Bull. Malaysian Math. Soc. 12 (1989) 2528.

[3] J.A. Gallian, A dynamic survey of graph labeling, Electron. J. Combin., 19 (2016).

[4] F. Harary, Sum graphs and difference graphs, Congr. Numer. 72 (1990) 101108.

[5] F. Harary, Sum graphs over all the integers. Discrete Math. 124 (1994), 99105.

[6] N. Hartsfield, W.F. Smyth, A family of sparse graphs of large sum number, Discrete Math. 141 (1995) 163171.

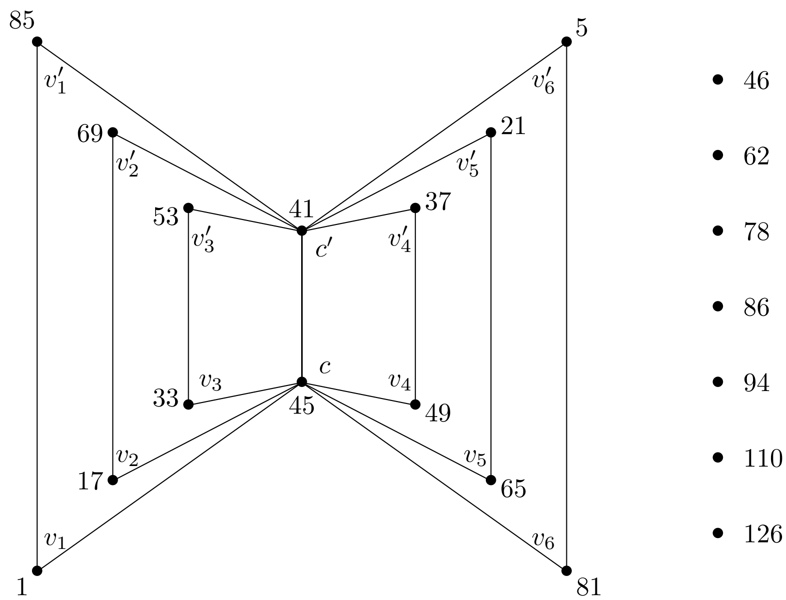

Figure 2: A Δ-optimum exclusive sum labeling for Book graph B_6

[7] M. Miller, D. Patel, J. Ryan, K. A. Sugeng, Slamin, M. Tuga, Exclusive sum labeling of graphs, J. Combin. Math. Combin. Comput., 55, (2005), 137 - 148.

[8] M. Miller, J. Ryan, Slamin, K. Sugeug, and M. Tuga, Open problems in exclusive sum graph labeling, preprint.

[9] M. Miller, J.F. Ryan, W.F. Smyth, The sum number of the cocktail party graph, Bull. Inst. Combin. Appl. 22 (1998) 7990.

[10] J. Ryan, Exclusive sum labeling of graphs: A survey, AKCE J. Graphs. Combin.,6(1) (2009), 113126.

[11] Sheetal Thakare and M. Acharya, Exclusive sum labeling and exclusive sum number of Prism graph (D_n), accepted for publication in Ars Combinatoria.

[12] M. Tuga and M. Miller, Δ-optimum exclusive sum labeling of certain graphs with radius one, Proceedings of IJCCGGT 2003, Bandung, Indonesia, (2003), 216-225.

[13] M. Tuga, M. Miller, J. Ryan and Z. Ryj acek, Exclusive sum labelings of trees, J. Combin. Math. Combin. Comput., 55, (2005), 109-121.