Generalized Fractional integrals involving Eulerian Integral of H-Function

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Abstract

A large number of fractional integral formulas involving certain special functions have been presented. Here, in this paper, our aim at establishing three fractional integral formulas involving the products of the multivariable *H*-function by using generalized fractional integration operators given by Saigo and Maeda [M. Saigo, N. Maeda, Varna, Bulgaria, (1996), 386–400]. the present paper we evaluate a number of key Eulerian integrals involving the H- function of several variables. Our general Eulerian integral formulas are shown to provide the key formulae from which numerous other potentially useful results for various families of generalized hypergeometric functions of several variables can be derived. In this paper, we evaluate a class of MacRobert's integral associ-ated with the multivariable I-function defined by Nambisan et al [1],

Also using I-Function on 'Certatain class of eulerian integrals of multivariable generalized hypergeometric function' (for details of H-function ,see [2-6] [7], [8],Saxena and Nishimoto [9],H.S.P. Srivastava [10]

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1.INTRODUCTIONAND DEFINITIONS

The results provided here provided closed-form expression for numerous other potentially useful integrals not contained in the aforementioned works.

Let α, β *and* η be complex numbers,and let x ∈ R_+= (0, ∞) Follwing Saigo [6} Fractional integral (Re(α)>0 and derivative Re(α) < 0 of first kind of a function f(x) on R_+ are defined respectively in the forms:

$$I_{0,\ x}^{\alpha,\beta,\eta}(f) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\, 2F1\left(\alpha+\beta\ ,\ -\eta\ ;\ \alpha\ ;\ 1-\frac{t}{x}\right) f(t)dt\ ; \qquad \ldots(1.1) \qquad Re(\alpha)>0$$

$$\frac{d^n}{dt^n} I_{0,\ x}^{\alpha+n\ \beta-n\ ,\eta-n}(f)\ ,\ 0< Re(\alpha)+n<1 \qquad (n= 1, 2, 3, \ldots\ldots\ldots\ldots),$$

Where 2F1(a, b; c; z) is Gauss's hypergeometric function

Fractional integral Re(α)>0 and derivative Re(α) < 0 of second kind a function f(x) on R_+ are given by:

$$J_{x,\infty}^{\alpha,\beta,\eta} = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\, t^{\alpha-\beta}\, 2F_1\left(\alpha+\beta\ ,\ -\eta\ ;\ \alpha\ ;\ 1-\frac{t}{x}\right) f(t)dt\ ; \qquad \ldots.(1.2) \qquad Re(\alpha)>0$$

$$= (-1)^n \frac{d^n}{dt^n} j_{x,\ \infty}^{\alpha+n\ \beta-n\ ,\eta}(f)\ , \qquad 0< Re(\alpha)+n<1 \quad (n= 1, 2, 3, \ldots),$$

Let α, β,η and λ be complex numbers. Then there hold the following formulae. . the R.H.S. has a definite meaning

$$I_{0,\ x}^{\alpha,\beta,\eta}\ t^{\lambda} = \frac{\Gamma(1+\lambda)\Gamma(1+\lambda-\beta+\eta)}{\Gamma(1+\lambda-\beta)\Gamma(1+\lambda+\alpha+\eta)} x^{1-\beta} \qquad \ldots..(1.3)$$

provided that Re(α)> max [0, Re(β-η)] — 1, and

$$j_{0,\ x}^{\alpha,\beta,\eta}\ t^{\lambda} = \frac{\Gamma(\beta-\lambda)\Gamma(\eta-\lambda)}{\Gamma(-\lambda)\Gamma(\alpha+\beta+\eta-\lambda)} x^{\lambda-\beta} \qquad \ldots..(1.4)$$

Re (α)>0 and Re(λ) < min [Re (β) , Re (η)], Re(α)< 0, 0< Re(α) + n <1 , and

Re(λ)< min [Re(β)-n , Re(η)] , Where n is a positive integer.

The integral (1.2) is a special case of (1.1) where $\alpha^2 + \beta^2 = \lambda^{-1}$ which was already pointed out by Wille [40] moreover more general results than (1.2) can be found out in the literature ([18], [19])

Meijer's G-Function and its celebrated generalization the Fox's H-Function [3] are defined in terms of single Meillin Barnes type contour integrals involving quotients of Gamma Function.it is evidenced in the literature.such single Meillin Barnes type contour integrals are useful in finding the analytic solutions of various problems in nuclear and neutrino astrophysics [12].where as the voigt function K(x,y) L(x, y) of astrophysical spectroscopy and of the theory of neutron reactions are expressible as double mellian barnes contour integrals of this same class ([8], [16]).moreover ,the multivariable H-Function contain much more general function as its special cases like generalized Kampe-de-Feriet and generalized lauricella functions (Srivastava-Daust) [15]

(i) The H-Function Defined by Saxena and kumbhat [21] is an extension of Fox's H-Function on specializing the parameters , H-Function can be reduced to almost all the known special function as well as unknown

The Fox's H-Function of one variable is defined and represented in this Paper as follows [see Srivastava et al [22] ,pp 11-13]

$$H[x] = H_{P,Q}^{M,N}\left[x \Big/ \begin{matrix} (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q} \end{matrix} \right] = \frac{1}{2\pi\omega} \int_{\theta=N-1} \theta(\xi) x^{\xi} d\xi \quad \text{....(1.5)}$$

$$\theta(\xi) = \frac{\prod_{i=1}^{n} \Gamma b_j - \beta_j \xi \prod_{j=1}^{N} \Gamma 1 - a_j - \alpha_j \xi}{\prod_{i=M=1}^{Q} \Gamma 1 - b_j + \beta_j \xi \prod_{j=N+1}^{P} \Gamma a_j - \alpha_j \xi}$$

For condition of the H-Function of one variable (1.5) and on the contour L we refer to srivastava et al [22]

The multivariable *H*-Function, introduced by Srivastava and Panda (see, [2] and [3]), is an extension of the multivariable *G*-function. This multivariable *H*-Function includes Fox's *H*-function, Meijer's *G*-function, the generalized Lauricella function of Srivastava and Daoust (see [23]), Apell function, the Whittaker function and so on. The multivariable *H*-function is defined and represented in the following manner

Fractional calculus which are derivatives and integrals of arbitrary (real and complex) orders have found many applications in a variety of fields ranging from natural science to social science. In recent years, it has turned out that many phenomena in engineering, physics, chemistry and other sciences can be described very successfully by means of models using mathematical tools deduced from fractional calculus. For example, the nonlinear oscillation of earthquakes can be modeled with fractional derivatives and the fluid-dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow. Fractional derivatives are also used in modeling many chemical processes, mathematical biology and many other problems in physics and engineering (see, *e.g.*, [31]–[29], [33], [34]).

Under various fractional calculus operators, the computations of image formulas for special functions of one or more variables are important from the point of view of the usefulness of these results in the evaluation of generalized integrals and the solution of differential and integral equations (see, *e.g.*,[12] , [13] , [17] , [27], [28],[32] and [35] (and so on). Motivated essentially by diverse applications of fractional calculus we establish two image formulas for the product of multivariable *H*-function and general class of polynomials involving left and right sided fractional integral operators of Saigo-Meada [11]. By virtue of the unified nature of our results, a large number of new and known results involving Saigo, Riemann-Liouville and Erd´elyi-Kober fractional integral operators and several special functions are shown to follow as special cases of our main results.

2.Main Results

Apply I- Function on The General Eulerian Integral of Multivariable H-Function. In this section, we establish three Result involving the products of one/two/ r-variable of *H*-function.

RESULT -1

$$I_{0,x}^{\alpha,\beta,\eta}\left\{t^{\lambda}\int_0^x t^{p-1}(x-t)^{\sigma-1}\ H_{P,Q}^{M,N}\left[zt^{u}(n-t)^{v}\left|\begin{matrix}(ap\ \ AP)\\ \vdots\\ (bQ.\ \ BQ)\end{matrix}\right.\right]dt\right]$$

$$=\frac{\lceil 1-(-\lambda)+\lceil 1-(\beta-\lambda-\eta)}{\lceil 1-(-\lambda+\beta)+\lceil 1-(-\lambda-\alpha-\eta)}$$

$$x^{\rho+\sigma+\lambda-\beta-1}\ H_{P+4\ ,Q+3}^{M,\ \ N+4}\left[zx^{(u+v)}\left|\begin{matrix}(1-\rho,\mu)\ (1-\sigma,v)\ (-\lambda,0)\ (\beta-\lambda-\eta,0)(ap,AP)\\ \vdots\\ (bQ,BQ)\ (1-\rho-\sigma,u+\rho)\ (-\lambda+\beta,0)(-\lambda-\alpha-\eta,0)\end{matrix}\right.\right]$$

……(1.6)

Provided (in addition to the appropriate convergence and existence conditions)that

$$\mu, \eta, \delta, \upsilon , a, b, c \ \lambda , \sigma > 0 \qquad \mathrm{Re}(a) > 0 \ \ \mathrm{Re}(\beta) > 0$$

RESULT -2

$$J_{x,\infty}^{\alpha,\beta,\eta}\left\{t^{\lambda}\int_{a}^{b}\{(t-a)(b-t)\}^{-\frac{1}{2}}\right.$$

$$H\left[\left\{\frac{(t-a)(b-t)}{A^{2}(t-a)+B^{2}(b-t)}\right\}^{v_1}\left\{\frac{(t-a)(b-t)}{A^{2}(t-a)+B^{2}(b-t)}\right\}^{v_2}\right]$$

$$=\sqrt{\pi}\ \frac{x^{\lambda-\beta}\ \lceil\beta-\lambda\ \lceil\eta-\lambda}{\lceil-\lambda\ \lceil\alpha+\beta+\eta-\lambda}\ \ H_{p+1,q+1\ p_1q_1,p_2,q_2}^{0\ n+1:m_1n_1,m_2,n_2}$$

$$\left[\begin{matrix}((b-a)/(B+A)^{2})^{v_1}\\((b-a)/(B+A)^{2})^{v_2}\end{matrix}\ \middle|\ \begin{matrix}\left(\frac{1}{2}.v_1v_2\right)\left(a_j\ \alpha_j^{1},\alpha_j^{2}\right)_{1,p}\left(c_j^{1}r_j^{1}\right)_{1,p_1}\left(c_j^{2}r_j^{2}\right)_{1,p_2}\\ \vdots \\ (0.v_1v_2)\left(b_j\beta_j^{1}.\beta_j^{2}\right)_{1,q}\left(d_j^{1}.\delta_j^{1}\right)_{1,q_1}\left(d_j^{2}.\delta_j^{2}\right)_{1,q_2}\end{matrix}\right] \quad \ldots.(1.7)$$

Provided (in addition to the appropriate convergence and existence conditions)that

$$\mu, \eta, \delta, \upsilon , a, b, c \ \lambda , \sigma > 0 \qquad \mathrm{Re}(a) > 0 \ \ \mathrm{Re}(\beta) > 0$$

RESULT -3

$$J_{x,\infty}^{\alpha,\beta,\eta}\left\{t^{\lambda}\int_{a}^{b}(t-a)^{\alpha-1}\ (b-t)^{\beta-1}\ (ut+v)^{\gamma}\ \ H[w_1x^{\rho_1}\ldots\ldots w_rx^{\rho_r}]\right\}$$

$$=\frac{\lceil\beta-\lambda\ \ \lceil\eta-\lambda}{\lceil-\lambda\ \ \lceil\alpha+\beta+\eta-\lambda}\ \ (b-a)^{\alpha+\beta-1}\ \frac{\lceil\alpha\ \lceil\beta}{\lceil\alpha+\beta}\ (au+v)^{\gamma}\ \sum_{m=0}^{\infty}\binom{r}{m}\left\{\frac{u(t-a)}{au+v}\right\}^{m}$$

$$H_{p,q\,;\ p_1,q_1,\ldots\ldots p_r,q_r}^{0,m\,;\,m_1,n_1\ldots\ldots m_r,n_r}$$

$$\left[\begin{array}{c|c} w_1 x^{\rho_1} & (aj,\alpha_j^i \ldots\ldots \alpha_j^r)_{1,p}\ (c_j^1\ r_j^1)_{1,p_1} \ldots\ldots (c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ \cdot & \\ \cdot & \\ \cdot & \\ w_r x^{\rho_r} & (b_j\ \beta_j^1 \ldots\ldots \beta_j^r)_{1,q}\ (d_j^1 .\delta_j^1)_{1,q_1} \ldots\ldots\ (d_j^{(r)}.\delta_j^{(r)})_{1,q_r} \end{array}\right] \quad \ldots\ldots.(1.8)$$

Provided (in addition to the appropriate convergence and existence conditions)that

$$\mu, \eta, \delta, \upsilon , a, b, c\ \ \lambda , \sigma > 0 \quad \mathrm{Re}(a) > 0\ \ \mathrm{Re}(\beta) > 0$$

Proof :- Result -1

{∵ *To Changing order of integration* then using Euler's first integral (Beta function) formula . and apply the formula (1.3). We interpret the resulting Mellin-Barnes contour integrals as a multivariable H- function, we get the desired result.

Proof :- Result –2

$Put\ \ t - a = (b - a)y \Rightarrow t = a + (b - a)y$

$then\ \ at = (b - a)dy \quad b - t = (b - a)(1 - y)$

then using Euler's first integral (Beta function) formula.and apply the formula (1.4) .

Now interpret the resulting Mellin-Barnes Contour Integral as a H-function of two-variables we get required result.

Proof :- Result -3

Now using the binomial expansions and apply the formula (1.4). using the mellian Barnes Contour Integral as a H-function of r-variables we get the desired result.

3.Conclusion

Here we presented two very generalized and unified theorems associated with the generalized fractional integral operators given by Saigo-Maeda. The main fractional integrals whose integrands being the products of multivariable *H*-functions The main results may find potentially useful applications in a variety of areas.In this paper we have evaluated a general Eulerian integral involving the multivariable I-function defined by Nambisan et al [1], a class of polynomials of several variables and the generalized incomplete hypergeometric function. The integral established in this paper is of very general nature as it contains multivariable I-function, which is a general function of several variables studied so far. Thus, the integral established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained

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