Solutions to the Nonlinear Fractional Boundary Value Problem

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Abstract: *The study of several positive solutions to a nonlinear fractional differential equation with boundary conditions is the focus of this article. The nonlinear term provides the positive continuous function. Through the use of Schauder's and Avery-Peterson's FPT, we are capable of to acquire the necessary conditions for several results of positive solution to the boundary value problems under consideration. In order to enhance the demonstrations, we included examples.*

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1. Introduction

Fractional calculus is an extension of classic calculus that allows the definition of derivatives and integrals for any real order. Fractional calculus has important applications in both pure mathematics and a number of practical applications. Fractional calculus has drawn a lot of attention due to its numerous applications in a wide range of sciences, including population dynamics, physics, chemistry, biophysics, control theory, capacitor theory, signal processing, and electromagnetics, among many others. Fractional differential equations are better at modelling phenomena than classical ones, and this is especially true for fractal theory, chaos, and bioengineering [13]. In the field of studying the existence and uniqueness of solutions to the boundary value problems of fractional derivatives, an extensive amount of research has been conducted and is available in [2,5,11,16,20]. Furthermore, recent studies have shown that fractional differential equations are a more useful tool for characterizing the dynamics of a wide range of systems. For more details, see [7,12,14,15,17-19].

This article aims to investigate the necessary conditions that lead to the presence of positive solutions for nonlinear fractional order derivative, together with the boundary conditions such as:

$$D^{\alpha} y(t) = f(t, y) = 0, \quad t \in (0,1),$$
(1)

$$y(0) = y(1) = 0,$$

where $f: [0,1] \times [0,\infty) \to [0,\infty)$ is continuous, $1 < \alpha \leq 2$ and D^{α} represents the α^{th} order R-L type differential operator.

In a cone in a Banach space, we use Schauder's FPT [9] and Avery-Peterson's FPT [3] as the main tool for FBVP.

Several researchers have used FPTs to show the existence of positive solutions for nonlinear fractional differential equations (FDEs) in the domains of ordinary differential equations, difference equations, and time-scale dynamics equations (see references [1,4] for more details).

In [2], B. Ahmad and J. J. Nieto used Leray-Schauder's FPT of nonlinear alternative to condense the mapping principle for nonlinear FBVP and demonstrate the necessary condition for existence and uniqueness of nontrivial solutions,

$$D^{\alpha} y(t) = f(t, y), \qquad t \in (0, 1),$$

with boundary conditions,

$$D^{\alpha-2} y(0) = \gamma_0 D^{\alpha-2} y(T),$$

$$D^{\alpha-1} y(0) = \mu_0 D^{\alpha-1} y(T),$$

where $\mu_0, \gamma_0 \neq 1, 1 < \alpha \leq 2$.

In [8], C. Goodrich has demonstrated the following BVP having at least three solutions,

$$D^{\alpha} y(t) = f(t, y) = 0, \quad t \in (0,1),$$

and

$$y^{i}(0) = 0, \qquad i = 0,1,2, \dots, n-2,$$

$$D^{\gamma} y(1) = 0, \qquad 2 \leq \gamma \leq n-2,$$

where $n-1 < \alpha \leq n,\ n > 3,\ n \in N,\ f: i \times [0,\infty) \to [0,\infty)$ is a continuous function.

In [10], E. R. Kaufmann and E. Mboumi used Krasnosel'skii FPT and Leggett-William's FPT to show that there are multiple positive solutions for the nonlinear FBVP,

$$D^{\alpha} y(t) + a(t) f(y) = 0, \qquad t \in (0,1), \qquad \alpha \in (1,2],$$

with

$$y(0) = 0, y'(1) = 0.$$

Our motivation comes from the work done in [2,6,8,10,21,22].

There are four sections. The introduction is in Section 1. In Section 2, we cover several kinds of fundamental definitions, concepts, lemmas, and theorems related to FDEs and concluded two well-known fixed-point theorems. We construct the main results in Section 3 that provide sufficient conditions to ensure the existence of multiple positive solutions to the FBVP (1) using Schauder's and

Avery-Peterson's FPT. We present some examples to support our results. The conclusion is presented in Section 4.

2. Basic Definitions and Concepts

2.1. Definition [12]: If α is a real number satisfying $\alpha \in (n - 1, n]$ and n is a positive integer, then the α^{th} R-L fractional integral of $y(t)$: $[0,\infty] \to R$ may be expressed as I^{α},

$$I^{\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \, y(s)ds \,,$$

provided that the RHS converges.

2.2. Definition [12]: Let n be a positive integer and let α be a real number such that $\alpha \in (n - 1, n]$, α^{th} R − L fractional derivative of $y(t)$: $[0,\infty] \to \mathcal{R}$ is represented by $D^{\alpha}y(t)$,

$$D^{\alpha}y(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^1 (t - s)^{n-\alpha-1} y(s)ds,$$

provided that the RHS converges.

2.3. Definition: Let $\boldsymbol{B}$ represent a Banach space over $\mathcal{R}$. If

(i) $\mu u + \eta v \in \mathcal{P}$, $\mu,\eta \geq 0$ $\forall$ $u,v \in \mathcal{P}$, and

(ii) $u \in \mathcal{P}, -u \in \mathcal{P} \Longrightarrow u = 0$, then $\mathcal{P}$ is a closed nonempty subset of $\boldsymbol{B}$.

2.4. Definition: $\Phi : \mathcal{P} \to \mathcal{R}_+$ which is nonnegative continuous function on a cone $\mathcal{P}$ of a real Banach space $\boldsymbol{B}$ is said to be nonnegative continuous concave functional if

$$\Phi(tu + (1 - t)v) \geq t\Phi(u) + (1 - t)\Phi(v) \quad \forall\, u,v \in \mathcal{P}, t \in [0,1].$$

Similarly, $\phi : \mathcal{P} \to \mathcal{R}_+$ which is a nonnegative continuous function on a cone $\mathcal{P}$ of a real Banach space $\boldsymbol{B}$ is said to be nonnegative convex functional if

$$\phi(tu + (1 - t)v) \leq t\phi(u) + (1 - t)\phi(v) \;\forall\, u,v \in \mathcal{P}, t \in [0,1].$$

2.5. Definition: An operator is known to as being completely continuous if it is continuous and translates bounded sets into precompact sets.

2.6. Lemma [9]: For $D^{\alpha}\, y(t) = 0$ along with $n - 1 < \alpha \leq n\,, n > 1$, then the general solution $y(t)$ is given by

$$y(t) = e_1 t^{\alpha-1} + e_2 t^{\alpha-2} + \ldots + e_n t^{\alpha-n}\,, \qquad e_i \in \mathcal{R}, \qquad i = 1,2,\ldots,n.$$

2.7. Lemma [9]. Following equality holds for $y(t)$, for assumed $\alpha > 0$,

$$I^{\alpha} \; D^{\alpha} \; y(t) = y(t) + e_1 t^{\alpha-1} + e_2 t^{\alpha-2} + \ldots + e_n t^{\alpha-n} \,,$$

where $e_i \in \mathcal{R}$, $i = 1,2,\ldots,n$.

2.8. Theorem: (Schauder's FPT [9]) Suppose that $\boldsymbol{B}$ is a Banach space, and let Ω be a non-empty, closed, bounded, convex subset of $\boldsymbol{B}$. Let $T : \Omega \rightarrow \Omega$ be an operator that is completely continuous. As a consequence, T has at least one fixed point in.

We will employ the notations mentioned below in accordance with Avery and Peterson [3]. In this case, we will examine two nonnegative convex functionals, ϕ *and* Θ, as well as a nonnegative. Let us consider two nonnegative convex functionals, ϕ and Θ and a nonnegative continuous concave functional be Φ on a cone $\mathcal{P}$, and let a nonnegative continuous functional on cone $\mathcal{P}$ be ψ. Thus, for the positive numbers $a_1, a_{2,}\, a_3$ and a_4, we set the following:

$$\mathcal{P}(\phi, a_4) = \{y \in \mathcal{P}: \phi(y) < a_4\};$$

$$\overline{P(\phi, a_4)} = \{y \in \mathcal{P}: \phi(y) \leq a_4\};$$

$$\mathcal{P}(\phi, \Phi, a_2, a_4) = \{y \in \mathcal{P}: a_2 \leq \Phi(y), \phi(y) \leq a_4\};$$

$$\mathcal{P}(\phi, \Theta, \Phi, a_2, a_3, a_4) = \{y \in \mathcal{P}: a_2 \leq \Phi(y), \Theta(y) \leq a_3, \phi(y) \leq a_4\};$$

$$\mathcal{R}(\phi, \psi, a_1, a_4) = \{y \in \mathcal{P}: a_1 \leq \psi(y), \phi(y) \leq a_4\}.$$

2.9. Theorem: (Avery and Peterson's FPT [3]) Consider **B** be a real Banach space and $\mathcal{P}$ be a cone in **B**. Let on cone $\mathcal{P}$, ϕ and Θ be nonnegative continuous convex functionals, assume that Φ be a nonnegative continuous concave functional on cone $\mathcal{P}$, and let ψ be a nonnegative continuous functional on $\mathcal{P}$ in a real Banach space **B** satisfying $\psi(ky) \leq k\psi(y)$ for $0 \leq k \leq 1$, that is, for positive numbers $\overline{N}$ and a_4,

$$\Phi(y) \leq \psi(y) \text{ and } ||y|| \leq \bar{N}\phi(y) \ \forall\, y \in \mathcal{P}(\phi, a_4).$$

Let $T: \overline{\mathcal{P}(\phi, a_4)} \rightarrow \overline{\mathcal{P}(\phi, a_4)}$ is completely continuous. Let us assume that there exist some constants $a_1, a_2,$ *and* a_3 with $a_1 < a_2$ such that

$(B1)$: $\{y \in \mathcal{P}(\phi, \Theta, \Phi, a_2, a_3, a_4): \Phi(y) > a_2\} \neq 0$ and $\Phi(T\,y) > a_2$ for $y \in \mathcal{P}(\phi, \Theta, \Phi, a_2, a_3, a_4)$;

$(B2)$: $\Phi(T\,y) > a_2$ for $y \in \mathcal{P}(\phi, \Phi, a_2, a_4)$ with $\Theta(T\,y) > a_3$;

$(B3)$: $0 \notin \mathcal{R}(\phi, \psi, a_1, a_4)$ and $\psi(T\,y) < a_1$ for $y \in \mathcal{R}(\phi, \psi, a_1\, a_4)$ with $\psi(y) = a_1$.

Then the operator $T: \overline{\mathcal{P}(\phi, a_4)} \rightarrow \overline{\mathcal{P}(\phi, a_4)}$ has atleast three fixed points $y_1, y_2, y_3 \in \mathcal{P}(\phi, a_4)$, such that $\phi(y_i) \leq a_4$, where $i = 1, 2, 3,$ $a_2 < \Phi(y_1),$ $a_1 < \psi(y_1), \Phi(y_2) < a_2,$ *and* $\psi(y_3) < a_1$.

3. Main Result

We define a cone $\mathcal{P} \subset \boldsymbol{B}$ by

$$\mathcal{P} = \{y \in \boldsymbol{B}: y(t) \geq 0, 0 \leq t \leq 1\}, \tag{2}$$

a Banach space $\boldsymbol{B} = (C[0,1], ||.||)$, with the norm $\left||y|\right| = \max_{0\leq t\leq 1} |y(t)|$.

3.1. **Lemma [21]:** For , $y \in C[0,1]$ and $\alpha \in (0,2]$ then the unique solution for FBVP

$$D^{\alpha}\, y(t) + u(t) = 0, \qquad t \in (0,1) \tag{3}$$

$$y(0) = y(1) = 0, \tag{4}$$

is given by

$$y(t) = \int_0^1 G(t,s)u(s)ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases}[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ [t(1-s)]^{\alpha-1}, & 0 \le t \le s \le 1.\end{cases} \tag{5}$$

3.2. **Lemma [5]:** $G(t,s)$ defined in (5), satisfying the conditions:

(i) For $(t,s) \in (0,1)$, $0 < G(t,s)$,

(ii) $\min_{\frac{1}{3}\le t\le\frac{2}{3}} G(t,s) \ge \gamma(s) \max_{0\le t\le 1} G(t,s) = \gamma(s)\ G(s,s)$, for $s \in (0,1)$.

Where $\gamma \in C(0,1)$.

The FBVP (1) has the solution by Lemma 3.1,

$$u(t) = \int_0^1 G(t,s)f(s,y)ds, \tag{6}$$

the operator $T : \mathcal{P} \to \mathcal{P}$ is defined by

$$T\,y(t) = \int_0^1 G(t,s)f(s,y(s))ds. \tag{7}$$

We employ the following notations for our convenience:

$$U = \frac{1}{\int_0^1 G(s,s)ds}, \quad V = \int_{1/3}^{2/3} \frac{1}{\gamma(s)G(s,s)ds} \tag{8}$$

3.3. **Lemma [5]:** Suppose that $f: [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function. Then, the operator T that is provided in equation (7) is continuous.

3.4. **Theorem:** For a continuous function $f: [0,1] \times [0,\infty) \to [0,\infty)$ that satisfies the Lipschitz condition in y, let's suppose that $k \in (0,U)$ exists such that, for $(t,y_1),(t,y_2) \in [0,1] \times \mathcal{R}$,

$$|f(t,y_1) - f(t,y_2)| \le k|y_1 - y_2|.$$

Therefore, the FBVP (1) has a unique solution. Moreover, there are no nontrivial solutions to FBVP (1) if $f(t,0) \equiv 0$.

Proof: The function $T: \mathcal{P} \to \mathcal{P}$, shown in equation (7),

$$T\,y(t) = \int_0^1 G(t,s)f(s,y)ds, \quad y \in \boldsymbol{B}.$$

The Green's function $G(t,s)$ is provided in equation (5). It is clear that if and only if $y(t)$ is a fixed point of T, then $y(t)$ provides the FBVP (1). T is clearly completely continuous.

We have for any $y_1, y_2 \in \boldsymbol{B}$ and $t \in [0,1]$,

$$|(T\,y_1 - T\,y_2)(t)| = \int_0^1 |\,G(t,s)[f(s,y_1) - f(s,y_2)]ds|$$

$$\leq \int_0^1 G(t,s)|f(s,y_1) - f(s,y_2)|ds$$

$$\leq \int_0^1 G(s,s)k\,|y_1 - y_2|ds$$

$$\leq k\,||y_1 - y_2|| \int_0^1 G(s,s)ds$$

$$\leq k\frac{1}{U}||y_1 - y_2||$$

$$\leq ||y_1 - y_2||.$$

Thus T is a contraction mapping. Now, operator T has a unique fixed point in **B** according to the Banach contraction mapping principle. As a result, the FBVP (1) has a unique solution. If $f(t,0) \equiv 0$ on $0 \leq t \leq 1$, then $y(t) \equiv 0$ is a solution to the FBVP (1). The uniqueness of solutions for FBVP (1) shows that there are no nontrivial solutions. The proof of the Theorem (3.4) is concluded.

3.5. **Theorem:** Assume that the function f is continuous, $f: [0,1] \times [0,\infty) \to [0,\infty)$ and $f(t,0) \not\equiv 0$ for $0 \leq t \leq 1$, for

$$\lim_{|y|\to\infty} \frac{|f(t,y)|}{|y|} = 0, \tag{9}$$

the FBVP (1), then has at least one nontrivial solution.

Proof: Equation (12) of Theorem (3.5), for every $0 \leq t \leq 1$ and y with $|y| \geq r_1$, there exists a $r_1 > 0$ such that $|f(t,y)| \leq U|y|$. It is certainly given for f that there is a constant $n_1 > 0$ such that, on $[0,1] \times [-r_1, r_1]$, $|f(t,y)| \leq n_1$. Assuming that $r_2 = \max\left\{r_1, \frac{n_1}{U}\right\}$, then, on $[0,1] \times [-r_2, r_2]$,

$$|f(t,y)| \leq Ur_2,$$

where U is defined in (9). We choose

$$\Omega_1 = \{y \in \boldsymbol{B}: ||y|| \leq r_2\}$$

to be a closed, convex and bounded set in $\boldsymbol{B}$. Next, for any $y \in \Omega_1$, we obtain

$$|(T\,y)(t)| = \left|\int_0^1 G(t,s)f(s,y)ds\right|$$

$$\leq \int_0^1 G(t,s)|f(s,y)|ds$$

$$\leq Ur_2 \int_0^1 G(s,s)ds$$

$$\leq r_2.$$

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It follows that $T(\Omega_1) \subset \Omega_1$. Then, T has at least one fixed point in Ω_1, according to the Theorem (2.8). Given that $f(t,0) \not\equiv 0$ on [0,1], it is obvious that $y(t) \equiv 0$ is not a fixed point of operator T. Accordingly, FBVP (1) has at least one nontrivial solution. Thus, the Theorem (3.5) is proved.

3.6. **Example:** With $f(t,y) = \mu(cosy + e^{2t^2})$, $\forall \mu \in [0,U]$, assume the FBVP (1).

$$|f(t,y_1) - f(t,y_2)| \leq \mu|y_1 - y_2|$$

for any $(t,y_1),(t,y_2) \in [0,1) \times \mathcal{R}$. Theorem 3.5 states that there is only one solution to this problem. Since $f(t,0) \not\equiv 0$, the solution is also nontrivial.

Now we will then determine whether FBVP (1) has at least three positive solutions.
A sub-cone $\mathcal{P}^*$ is defined for the cone $\mathcal{P}$ in (3),
where

$$\mathcal{P}^* = \{y \in \boldsymbol{B}: \min_{t\in[\frac{1}{3},\frac{2}{3}]} y(t) \geq \gamma||y||\}.$$

(10)

3.7. **Theorem:** Let $f: [0,1] \times [0,\infty) \to [0,\infty)$ be continuous, and let $0 < a < b < c = \frac{b}{\gamma} \leq d$ be positive constants. Then,

(D1): $f(t,y) < U\,a,\ \forall\,(t,y) \in [0,1] \times [0,a]$,

(D2): $f(t,y) \geq V\,b,\ \forall\,(t,y) \in [\frac{1}{3},\frac{2}{3}] \times [b,d]$,

(D3): $f(t,y) \leq U\,d, \forall\,(t,y) \in [0,1] \times [0,d]$,

Thus, there are at least three positive solutions to the FBVP (1), y_1, y_2, and $y_3 \in \mathcal{P}^*$ and $||y_i|| \leq d$, for $i = 1,2,3$, such that $\psi(y_3) < a < \psi(y_2)$ and $\Phi(y_2) < b < \Phi(y_1)$.

Proof: A continuous concave functional Φ on $\mathcal{P}^*$ that is nonnegative is defined as follows:

$$\Phi(y) = \min_{t\in[\frac{1}{3},\frac{2}{3}]} |y(t)|.$$

Then we get

$$\Phi(y) \leq ||y||.$$

A nonnegative continuous function ψ on cone $\mathcal{P}^*$ is defined by

$$\psi(y) = ||y||.$$

Let two nonnegative continuous convex functionals on a cone $\mathcal{P}^*$ are ϕ and Θ,

$$\Theta(y) = \phi(y) = ||y||.$$

Consequently, $\psi(ry) = ||ry|| \leq |r|\,||y|| = |r|\,\psi(y) = r\,\psi(y)$, since $r \in [0,1]$,

$$\Phi(y) = \min_{t\in[\frac{1}{3},\frac{2}{3}]} |y(t)| \leq \big||y|\big| = \psi(y)$$

Additionally, $\overline{N} \geq 1$ may be found such that $||y|| = \phi(y) \leq \overline{N}\phi\,(y)\ \forall\, y \in \overline{\mathcal{P}^*(\phi,d)}$.

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Next, we demonstrate that Theorem (2.9)'s criteria (B1)–(B3) are satisfied. Assume that $T: \boldsymbol{B} \to \boldsymbol{B}$ given in (10). $y(t)$ is a solution of the FBVP (1), if and only if, $y(t)$ is a fixed point of T in $\boldsymbol{B}$. Lemma (3.2) states that $G(t,s) \geq 0$ for $t,s \in [0,1]$, and $f: [0,1] \times [0,\infty) \to [0,\infty)$ implies that $T\big(y(t)\big) \geq 0$ for all $0 < t \leq 1$.

We can demonstrate $T: \overline{\mathcal{P}^*(\phi,d)} \to \overline{\mathcal{P}^*(\phi,d)}$ based on the Lemma (3.3). Let for $y \in \overline{\mathcal{P}^*(\phi,d)}$, $\phi(y) = ||y|| \leq d$ for all $t \leq 0$ and $0 \leq y \leq d$. Then, by (D3),

$$||T\,y|| = \max_{t\in[0,1]} |(T\,y)(t)| = \max_{t\in[0,1]} \left| \int_0^1 G(t,s)\, f(s,y)ds \right|$$
$$\leq f(s,y) \int_0^1 G(s,s)ds$$
$$\leq d.$$

This means that $T: \overline{\mathcal{P}^*(\phi,d)} \to \overline{\mathcal{P}^*(\phi,d)}$. The operator $T: \overline{\mathcal{P}^*(\phi,d)} \to \overline{\mathcal{P}^*(\phi,d)}$ must now be shown to be completely continuous. Given that $G(t,s)$ and $f(t,y)$ for $(t,s) \in [0,1] \times [0,1]$ are continuous, T must be continuous on cone a $\mathcal{P}^*$.

Assume that

$$\overline{\mathcal{P}^*}_d = \{y \in \mathcal{P}^*: ||y|| \leq d\},$$

for $d > 0$.

Following that, we construct $H_1 = \max_{\substack{0\leq t\leq 1,\\ 0\leq y\leq d}} f(s,y)$. As we have

$$|(T\,y)(t)| = \left| \int_0^1 G(t,s) f(s,y)ds \right| = H_1 \int_0^1 G(s,s)ds$$

$$\leq \frac{H_1}{U},$$

demonstrates that the function T has a uniform boundary on $\overline{\mathcal{P}^*}_d$. $G(t,s)$ is uniformly continuous on $[0,1] \times [0,1]$since it is continuous on that interval. Consequently, for each $\epsilon > 0$, there is a $\delta > 0$ such that, for any $y \in \overline{\mathcal{P}^*}_d$ *and* $t_1, t_2 \in [0,1]$, with $|t_1 - t_2| < \delta$, $|G(t_1,s) - G(t_2,s)| < \epsilon$. Then,

$$|(T\,y)(t_1) - (T\,y)(t_2)| \leq \int_0^1 |G(t_1,s) - G(t_2,s)|\, f(s,y(s))ds$$

$$\leq \epsilon\, H_1,$$

$T(\overline{\mathcal{P}^*}_d)$ is equicontinuous, as this implies. As a consequence, the operator $T(\overline{\mathcal{P}^*}_d)$ is relatively compact. T is completely continuous by an application of the Arzelà-Ascoli Theorem.
The function $y(t) = \frac{b+d}{2} = c \in \mathcal{P}^*(\phi,\Phi,\Theta,b,c,d)$and $\frac{\Phi(b+d)}{2} > b$, implies that

$\{y \in \mathcal{P}^*(\phi,\Phi,\Theta,b,c,d): \Phi(y) > b\} \neq 0$, for $y \in \mathcal{P}^*(\phi,\Phi,\Theta,b,c,d): \Phi(y) > b\}$, we have $b \leq y(t) \leq \frac{b}{\gamma}$ for $t \in [\frac{1}{3}, \frac{2}{3}]$. From assumption (D2),

$$\Phi(T\,y) = \min_{t\in[\frac{1}{3},\frac{2}{3}]} ||T(y)||$$

$$\geq \min_{t\in[\frac{1}{3},\frac{2}{3}]} \int_0^1 G(t,s)\, f(s,y)ds$$

$$\geq \int_0^1 \gamma(s)L(s)f(s,y(s))ds$$

$$> V\ b \int_{\frac{1}{3}}^{\frac{2}{3}} \gamma(s)G(s,s)ds$$

$$> b,$$

shows that (B1) of Theorem (2.9) has been satisfied, which implies $\Phi(T\ y) > b\ \forall\ y\ \in \{y \in P^*(\phi, \Phi, \Theta, b, \frac{b}{\gamma}, d)\}$. Additionally suppose that $y \in \mathcal{P}^*(\phi, \Phi, b, d)$ and that $\Theta(y) > c = \frac{b}{\gamma}$.

$$\Phi(T\ y) = \min_{t\in[\frac{1}{3},\frac{2}{3}]} (T\ y)(t) \geq \gamma\ ||T\ y|| = \gamma\Theta(T\ y) > \gamma c = b.$$

Furthermore, (D2) and (B2) of Theorem (2.9) are significantly related. Hence, (B2) of Theorem (2.9) satisfied.

The fact that $\phi(0) = 0 < b$ clearly shows that $\phi \in \mathcal{R}(\phi, \psi, a, d)$. Let y be a point in $\mathcal{R}(\phi, \psi, a, d)$ and suppose that $\psi(y) = ||y|| \leq a$. Then by (D1),

$$\psi(T\ y) = \max_{t\in[0,1]} |\int_0^1 G(t,s)\ f(s,y)ds\ |$$

$$< \int_0^1 G(s,s)f(s,y)ds$$

$$< U\ a \int_0^1 G(s,s)ds$$

$$< a.$$

Accordingly, condition (B3) of Theorem (2.9) is satisfied. Consequently, according to Theorem (2.9), the problem (1) has at least of three positive solutions, y_1, y_2, and y_3, together with $||y_i|| \leq d$, where $i = 1, 2, 3$, and $\psi(y_3) < a < \psi(y_2)$, $\Phi(y_2) < b < \Phi(y_1)$. This demonstrates the proof of the Theorem (3.7).

3.8. **Example:** Let $D^{\frac{3}{2}} y(t) + f(t,y) = 0,\ y(0) = y(1) = 0, \quad t \in (0,1),$ (11)

Where,

$$f(t,y) = \begin{cases} \frac{\sqrt{t}}{10} + 10y^2, & \forall\ y \leq 1, \\ 9 + \frac{\sqrt{t}}{10} + \frac{y}{2}, & \forall\ y > 1. \end{cases}$$

By simplification, we get $U \approx 2.25676,\ N \approx 13.6649$.

By choosing $a = \frac{1}{12}, b = \frac{1}{10}, c = 10, d = \frac{23}{2}$,

$$f(t,y) = \frac{\sqrt{t}}{10} + 10y^2 \leq 0.15929 \leq U\ a \approx 0.187311, \quad \forall\ (t,y) \in [0,1] \times \left[0, \frac{1}{12}\right],$$

$$f(t,y) = 9 + \frac{\sqrt{t}}{10} + \frac{y}{2} \geq 9.120711 \geq V\ b \approx 1.36649, \quad \forall\ (t,y) \in \left[\frac{1}{3}, \frac{2}{3}\right] \times \left[\frac{1}{10}, \frac{23}{2}\right],$$

$$f(t,y) = 9 + \frac{\sqrt{t}}{10} + \frac{y}{2} \leq 14.85 \leq Ud \approx 25.9624, \quad \forall\ (t,y) \in [0,1] \times \left[0, \frac{23}{2}\right].$$

The FBVP (11) thus has at least three positive solutions, y_1, y_2 *and* y_3, with $\psi(y_3) < \frac{1}{2} < \psi(y_2)$ and $\Phi(y_2) < \frac{1}{10} < \Phi(y_2)$, according to Lemma (2.9).

4. Conclusion

We came to the conclusion that there are multiple positive solutions of fractional differential equation (FDE) for the time frame chosen in this study using fixed point theorems (FPTs) due to Schauder's FPT and Avery-Peterson's FPT. We intend to demonstrate that employing these FPTs and determining the appropriate interval provides positive results. We also demonstrated the theoretical conclusions using examples. We conclude, in brief, that fixed point theories are ideal tools to deal with classical differential equations as well as fractional differential equations (FDEs).

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References

[1] R. P. Agarwal, D. O'Regan and P. J. Y. Wong; Positive solutions of Differential, Difference, and Integral Equations, Kluwer Academic Publishers, Boston, 1999.

[2] B. Ahmad, J. J. Nieto; Fixed Point Theory, Vol.13, No.2, pp.329- 336(2012).

[3] R. I. Avery and A. C. Peterson; Three positive fixed points of nonlinear operators on ordered Banach spaces, Comput. Math. Appl. 42 (2001), 313 - 505.

[4] R. I. Avery, J. M. Davis and J. Henderson; Three symmetric positive solutions for lid stone problems by a generalization of the Leggett-Williams Theorem, Electron. J. Diff. Eqns. 2000 (2000), No. 40, 1-15.

[5] Z. Bai; Z. Bai and H. Lu; Positive solutions for a boundary value problem of nonlinear fractional differential equation, J. Math. Anal. Appl. 311 (2005), 495-505.

[6] Z. Bai; On positive solutions of a nonlocal fractional boundary value problem, Nonlinear Analysis, Vol.72, pp.916-924(2010).

[7] S. Das; Functional Fractional Calculus for system Identification and Controls, Springer, New York, NY, USA, 2008.

[8] C. Goodrich; Existence of a positive solution to a class of fractional differential equations, Comput. Math. with Appl., Vol.59, pp.3889-3999 (2010).

[9] A. Granas and J. Dugundji; Fixed Point Theory, New York: SpringerVerlag, 2003.

[10] E. R. Kaufmann and E. Mboumi; Positive solutions of a boundary value problem for a nonlinear fractional differential equation, Electron. J. Qual. Theory Differ. Equ. 2008, No. 3, 1-11.

[11] R. A. Khan and M. Rehman; Existence of Multiple Positive Solutions for a General System of Fractional Differential Equations, Commun. Appl. Nonlinear Anal., Vol.18, pp.25-35 (2011).

[12] A. Kilbas, H. Srivastava and J. Trujillo; Theory and Applications of Fractional Differential Equations, North-Holland, Amsterdam (2006).

[13] R. L. Magin; Fractional Calculus in Bioengineering, Begell House Publishers, (2006).

[14] K. Miller, B. Ross; An Introduction to the Fractional Calculus and Fractional Differential Equation, Wiley, New York (1993).

[15] K. Oldham, J. Spanier; The Fractional Calculus, Academic Press, New York (1974).

[16] S. Padhi, J. R. Graef and S. Pati; Multiple positive solutions for a boundary value problem with nonlinear nonlocal Riemann-Stieltges integral boundary conditions, Fract Cal. Appl. Anal., Vol 21, No. 3 (2018), DOI: 10.1515/fca-2018-0038.

[17] I. Podlubny; Fractional Difference Equations, Academic Press, San Diego (1999).

[18] J. Sabatier, O. P. Agrawal and J. A. T. Machado, Eds.; Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering, Springer, Dordrecht, The Netherlands, 2007.

[19] S. Samko, A. Kilbas, O. Marichev; Fractional Integral and Derivative, Theory and Applications, Gordon and Breach, Yverdon (1993).

[20] K. Shah, H. Khalil and R. A. Khan; Investigation of positive solution to a coupled system of impulsive boundary value problems for nonlinear fractional order differential equations, Appl. Math. inf. Sci., Vol.77, pp.240-246(2015).

[21] K. Shah, Salman Zeb and R. A. Khan; Multiplicity Results of MultiPoint Boundary Value Problem of Nonlinear Fractional Differential Equations, Appl. Math. inf. Sci., Vol. 12 (2018), No. 4, 727-734(2018).

[22] S. Zhang; Positive solutions for boundary value problems of nonlinear fractional differential equations, Electron. J. Diff. Eqns. 2006 (2006), No. 36, 1-12.