Non-negative Solutions to the Nonlinear Fractional Boundary Value Problem

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Abstract: *This paper investigates the possibility of having a number of positive solutions for the type of nonlinear fractional differential equations. There are certain boundary conditions applied to this equation. The main objective is to identify the conditions that allow for the existence of these multiple positive solutions. In order to attain this, we use Avery-Peterson's fixed-point theorem and Schauder's fixed point theorem, obtaining sufficient condition for the solutions. In order to improve clarity, the article offers examples and demonstrations that support its conclusions. To simplify it briefly, the article uses fixed-point theorems (FPTs) to define the criteria necessary for the existence of multiple positive solutions in the nonlinear fractional differential equations under consideration, together with the corresponding boundary conditions.*

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1. Introduction

According to their nonintegral and generalized nature, fractional order differential equations are extremely important in many fields, such as heat transfer, quantum physics, elasticity, electric motors, control systems, and continuum mechanics. They differ from classical integer-order models in that they have the unique ability to describe memory and inherited characteristics of various materials and processes. References [9,11,20] include detailed study results, whereas [2-8, 12-18] go into further detail on basic theory, current developments, and real-world applications. Although the theory of BVPs for nonlinear fractional differential equations has progressed, there are still many unsolved questions that need to be investigated at.

In [4], Bai and L$\ddot{u}$ showed the existence of positive solutions to the FBVP,

$$D^{\alpha}\, y(t) + f(t, y(t)) = 0, \quad 0 < t < 1, 1 < \alpha \leq 2$$

$$y(0) = 0, \qquad y(1) = 0.$$

In [9], H. Gao and X. Han used the Krasnosel'skii FPT to show the existence of at least one positive solution and Leggett-William FPT to show the existence of at least three positive solutions of the nonlinear FBVP,

$D^{\alpha}\, y(t) + a(t)f(t, y(t)) = 0, \;\; t\, \epsilon\, (0,1), \;\; \alpha\, \epsilon\, (1,2]$

$y(0) = 0, \qquad y(1) = \sum_{i=1}^{\infty} \alpha_i y(\xi_i),$

where D^{α} is the standard R-L fractional derivative and $0 < \xi_i < 1, \;\; 0 \leq \; \alpha_i < \infty$.

In [15], C. F. Li et al. developed the existence and multiplicity results of positive solutions by using FPTs in nonlinear FBVP,

$D^{\alpha}\, y(t) + f(t, y(t)) = 0, \;\; t\, \epsilon\, (0,1), \;\; \alpha\, \epsilon\, (1,2]$

$y(0) = 0, \qquad D^{\beta}\, y(1) = \; aD^{\beta}\, y(\xi)$

where D^{α} stands for standard R-L fractional order differential operator.

The goal of this work is to examine whether the following kinds of nonlinear fractional differential equation have positive solutions, along with the corresponding boundary conditions:

$D^{\sigma}\, y(t) + h(t, y(t)) = 0, \;\; t\, \epsilon\, (0,1), \;\; \sigma\, \epsilon\, (1,2)$

$y(0) = 0, \;\; D^{\rho}\, y(1) = \; pD^{\rho}\, y(\zeta),$ (1)

WhereD^{σ} represents the σ-th order R-L type differential operator and $0 < \rho \leq 1, \;\; 1 < \sigma < 2, \;\; \sigma - \rho - 1 \geq 0, \;\; 0 \leq p \leq 1, \;\; \zeta < 1$.

This work appears to be divided into four sections. The introduction is provided in Section 1. The definitions, concepts, lemmas, and theorems related to fractional differential equations are explored in Section 2, and also two well-known FPTs are discussed in it. These significant findings are presented in Section 3. Section 4 contains a conclusion of the results.

2. Basic Definitions and Concepts

2.1. Definition [11]: Consider a positive integer n, and α be a real number satisfying $\alpha \in (n-1, n]$, then the $\alpha - th$ R-L fractional integral of $y(t)$: $[0, \infty] \; \rightarrow \mathcal{R}$, denoted by I^{α},

$$I^{\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^t (t \; - \; s)^{\alpha-1} \; y(s)ds \, ,$$

deliver that the RHS converges.

Similarly, the $\alpha - th$ R-L fractional derivative of $y(t)$: $[0, \infty] \; \rightarrow \; \mathcal{R}$, denoted by $D^{\alpha}y(t)$,

$$D^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^1 (t-s)^{n-\alpha-1}y(s)ds,$$

deliver that the RHS converges.

2.2. Definition: Consider the Banach space over $\mathcal{R}$ to be $\boldsymbol{B}$. $\mathcal{P}$, a closed nonempty subset of $\boldsymbol{B}$, is a cone if and only if the following are true:

(i) $\mu u + \eta v \in \mathcal{P},\ \mu,\eta \geq 0\ \forall\ u,v \in \mathcal{P}$, and

(ii) $u \in \mathcal{P}, -u \in \mathcal{P} \Longrightarrow u = 0$.

2.3. Definition: $\Phi : \mathcal{P} \to \mathcal{R}_+$, a continuous function on a cone $\mathcal{P}$, is said to be nonnegative continuous concave functional if

$$\Phi(tu + (1-t)v) \geq t\Phi(u) + (1-t)\Phi(v)\ \ \forall\, u,v \in \mathcal{P}, t \in [0,1].$$

Similarly, $\phi : \mathcal{P} \to \mathcal{R}_+$, a continuous function on a cone $\mathcal{P}$, is said to be nonnegative convex functional if

$$\phi(tu + (1-t)v) \leq t\phi(u) + (1-t)\phi(v)\ \forall\, u,v \in \mathcal{P}, t \in [0,1].$$

2.4. Definition: An operator shows completely continuity if that operator is continuous and transform a bounded set into precompact set.

2.5. Lemma [21]: For $D^{\alpha}\, y(t) = 0$, along with $n-1 < \alpha \leq n, n > 1$, then the general solution $y(t)$ is given by

$$y(t) = e_1 t^{\alpha-1} + e_2 t^{\alpha-2} + \ldots + e_n t^{\alpha-n}, \qquad e_i \in \mathcal{R}, \qquad i = 1,2,\ldots,n.$$

2.6. Lemma [21]. Following equality holds for $y(t)$, for assumed $\alpha > 0$,

$$I^{\alpha}\, D^{\alpha}\, y(t) = y(t) + e_1 t^{\alpha-1} + e_2 t^{\alpha-2} + \ldots + e_n t^{\alpha-n},$$

where $e_i \in \mathcal{R},\ i = 1,2,\ldots,n$.

2.7. Theorem: (Schauder's FPT [10]) Assume $\boldsymbol{B}$ be a Banach space and Ω be a subset of $\boldsymbol{B}$ which is a non-empty, bounded, convex and closed. Let a completely continuous operator be $T: \Omega \to \Omega$. Thus in Ω, T has at least a fixed point.

According to the Avery and Peterson [1], we will implement the notations listed below. Let us consider two nonnegative convex functionals, ϕ and Θ and a nonnegative continuous concave functional be Φ on a cone $\mathcal{P}$, and let a nonnegative continuous functional on cone $\mathcal{P}$ be ψ. Thus, for the positive constants a_1, a_2, a_3, and a_4, setting the following:

$\mathcal{P}(\phi, a_4) = \{y \in \mathcal{P}: \phi(y) < a_4\}$;

$\mathcal{P} = \{y \in \mathcal{P}: \phi(y) \leq a_4\}$;

$\mathcal{P}(\phi, \Phi, a_2, a_4) = \{y \in \mathcal{P}: a_2 \leq \Phi(y), \phi(y) \leq a_4\}$;

$\mathcal{P}(\phi, \Theta, \Phi, a_2, a_3, a_4) = \{y \in \mathcal{P}: a_2 \leq \Phi(y), \Theta(y) \leq a_3, \phi(y) \leq a_4\}$;

$\mathcal{R}(\phi, \psi, a_1, a_4) = \{y \in \mathcal{P}: a_1 \leq \psi(y), \phi(y) \leq a_4\}$.

2.8. Theorem: (Avery and Peterson's FPT [1]) Consider $\boldsymbol{B}$ be a real Banach space and $\mathcal{P}$ be a cone in $\boldsymbol{B}$. Convex functionals ϕ and Θ on cone $\mathcal{P}$ are nonnegative continuous functionals. Consider Φ to be a nonnegative continuous concave functional on cone $\mathcal{P}$. Let ψ be a nonnegative continuous functional that satisfies for $0 \leq k \leq 1$, $\psi(ky) \leq k\psi(y)$, i.e. for positive values $\overline{N}$ and a_4

$$\Phi(y) \leq \psi(y) \text{and } ||y|| \leq \overline{N}\phi(y)\ \forall\ y \in \overline{\mathcal{P}(\phi, a_4)}.$$

Assuming the completely continuous operator

$$T: \overline{\mathcal{P}(\phi, a_4)} \rightarrow \overline{\mathcal{P}(\phi, a_4)}.$$

Let us assume that there exist some constants $a_1, a_2, and\ a_3$ with $a_1 < a_2$ such that

$(B1)$: $\{y \in \mathcal{P}(\phi, \Theta, \Phi, a_2, a_3, a_4): \Phi(y) > a_2\} \neq 0$ and $\Phi(Ty) > a_2$ for $y \in \mathcal{P}(\phi, \Theta, \Phi, a_2, a_3, a_4)$;

$(B2)$: $\Phi(T\ y) > a_2$ for $y \in \mathcal{P}(\phi, \Phi, a_2, a_4)$ with $\Theta(T\ y) > a_3$;

$(B3)$: $0 \notin \mathcal{R}(\phi, \psi, a_1, a_4)$ and $\psi(T\ y) < a_1$ for $y \in \mathcal{R}(\phi, \psi, a_1\ a_4)$ with $\psi(y) = a_1$.

Then the operator $T: \overline{\mathcal{P}(\phi, a_4)} \rightarrow \overline{\mathcal{P}(\phi, a_4)}$ has atleast three fixed points $y_1, y_2, y_3 \in \mathcal{P}(\phi, a_4)$, such that $\phi(y_i) \leq a_4$, where $i = 1, 2, 3$, $a_2 < \Phi(y_1)$, $a_1 < \psi(y_1)$, $\Phi(y_2) < a_2$, $and\ \psi(y_3) < a_1$.

3. Main Result

To prove our results, with the norm $||y|| = \max_{0\leq t\leq 1} |y(t)|$, we define a Banach space $\boldsymbol{B} = (C[0,1], ||.||)$, and a cone $\mathcal{P} \subset \boldsymbol{B}$ by

$\mathcal{P} = \{y \in \boldsymbol{B}: y(t) \geq 0, 0 \leq t \leq 1\}$,

3.1. **Lemma [16]:** If $y \in C[0,1]$ and $\sigma \in (1,2)$, then for FBVP (1), the solution is

$$y(t) = \int_0^1 G(t,s)h(s,y)ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\sigma)}\begin{cases} At^{\sigma-1}(1-s)^{\sigma-\rho-1} - pAt^{\sigma-1}(\zeta-s)^{\sigma-\rho-1} - (t-s)^{\sigma-1}, & 0 \leq s \leq t \leq 1, \zeta \geq s, \\ At^{\sigma-1}(1-s)^{\sigma-\rho-1} - (t-s)^{\sigma-1}, & 0 \leq \zeta \leq s \leq t \leq 1, \\ At^{\sigma-1}(1-s)^{\sigma-\rho-1} - pAt^{\sigma-1}(\zeta-s)^{\sigma-\rho-1}, & 0 \leq t \leq s \leq 1, \zeta \leq 1, \\ At^{\sigma-1}(1-s)^{\sigma-\rho-1}, & 0 \leq t \leq s \leq 1. s \geq \zeta, \end{cases} \quad (2)$$

where $A = \frac{1}{(1-p\zeta^{\sigma-\rho-1})}$

3.2. **Lemma [16]:** In (2), $G(t,s) \geq 0$ is given that satisfies the following conditions:

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$$\frac{At^{\sigma-1}(1-s)^{\sigma-\rho-1}}{\Gamma(\sigma)} \geq G(t,s) \geq \frac{\rho t^{\sigma-1}s(1-s)^{\sigma-\rho-1}}{\Gamma(\sigma)} \forall\, (t,s) \in [0,1].$$

According to the Lemma (3.1), FBVP (1) has the solution,

$u(t) = \int_0^1 G(t,s)h(s,y)ds,$

and $T : \mathcal{P} \to \boldsymbol{B}$ is defined by

$$T\,y(t) = \int_0^1 G(t,s)h(s,y)ds. \tag{3}$$

We use some notations,

$$X = \frac{\Gamma(\sigma)(\sigma-\rho)}{A}, \quad J = \frac{\Gamma(\sigma)(\sigma-\rho)(\sigma-\rho+1)}{\rho\zeta^{\sigma-1}(1-\zeta)^{\sigma-\rho}(\sigma\zeta-\rho\zeta+1)} \tag{4}$$

3.3. **Theorem:** For a continuous function h, satisfies the Lipschitz condition in y, i.e. there exist a constant $V \in (0,X)$ such that

$$|h(t,y_1) - h(t,y_2)| \leq V|y_1 - y_2| \text{ for } (t,y_1),(t,y_2) \in [0,1] \times \mathcal{R}. \tag{5}$$

Then FBVP (1) has a unique solution. If, in addition, $h(t,0) \equiv 0$ then FBVP (1) has no nontrivial solutions.

Proof: Consider $T: \mathcal{P} \to \boldsymbol{B}$ given by (3) by where the Green's function $G(t,s)$ is given in (2). It is obvious that operator T is completely continuous and $y(t)$ is a solution of the FBVP (1) if and only if $y(t)$ is a fixed point of T in $\boldsymbol{B}$.

For any $y_1, y_2 \in \boldsymbol{B}$ and $t \in [0,1]$,

$$|(T\,y_1(t) - T\,y_2(t))| = \left|\int_0^1 G(t,s)[h(s,y_1) - h(s,y_2)]ds\right|$$

$$\leq \int_0^1 \frac{At^{\sigma-1}(1-s)^{\sigma-\rho-1}}{\Gamma(\sigma)} V\,|y_1(s) - y_2(s)|ds$$

$$\leq V\frac{1}{X}||y_1 - y_2||$$

$$\leq ||y_1 - y_2||.$$

Thus, the operator T shows a contraction mapping. As a result, T has a unique fixed point in $\boldsymbol{B}$ as according to the Banach contraction mapping principle. Thus, the FBVP (1) has unique solution. If $h(t,0) \equiv 0$ on $t \in [0,1]$, then obviously $y(t) \equiv 0$ is a solution of FBVP (1). By the uniqueness of solutions, FBVP (1) has no nontrivial solution. The proof of the Theorem (3.3) is now demonstrated.

3.4. **Theorem:** Let h be continuous. Suppose that,

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$$\lim_{|y|\to\infty} \frac{|f(t,y)|}{|y|} = 0 \tag{6}$$

and $h(t,0) \not\equiv 0$ on $t \in [0,1]$, Then problem (1) has at least a nontrivial solution.

Proof: By (6), there exist $b_1 > 0$ such that $|h(t,y)| \leq X\,|y|$ for any $0 \leq t \leq \infty$ and $|y| \geq b_1$. Clearly, h implies that there exist a constant $d > 0$ such that $|h(t, y)| \leq d$ on $[0,1] \times [-b_1, b_1]$. Consider $b_2 = max\,\{b_1, \frac{d}{X}\}$. Then

$$|h(t,y)| \leq Xb_2 \text{ on } [0,1] \times [-b_2, b_2].$$

Assuming that $\Omega = \{y \in \boldsymbol{B}: ||y|| \leq b_2\}$ be a bounded, closed, and convex set in $\boldsymbol{B}$. Then for any $y \in \Omega$, we have

$$|(T\,y)(t)| = \left|\int_0^1 G(t,s)h(s,y)ds\right|$$

$$\leq Xb_2 \int_0^1 G(t,s)h(s,y)ds$$

$$\leq Xb_2 \int_0^1 \frac{At^{\sigma-1}(1-s)^{\sigma-\rho-1}}{\Gamma(\sigma)}$$

$$\leq b_2.$$

Thus, $||T\,y|| \leq b_2$, that is $T(\Omega) \subset \Omega$. Then by the Schauder's FPT (2.7), in $\boldsymbol{\Omega}$, T has at least one fixed point. It is clear that $y(t) \equiv 0$ is not a fixed point of T because $h(t,0) \not\equiv 0$ on $[0,1]$. Hence, the problem (1) has at least a nontrivial solution. This demonstrates the proof of the Theorem (3.4).

3.5. **Example:** Assuming the FBVP (1) with $h(t,y) = e^{\frac{t}{3}} + \gamma\, tany$ where $\gamma \in (0, X)$. Then

$|h(t,y_1) - h(t,y_2)| \leq \gamma|y_1 - y_2|$ for $(t,y_1), (t,y_2) \in [0,1] \times \mathcal{R}$. Hence according to the Theorem (3.3), this problem has a unique solution. Also, clearly, (6) is satisfied. Hence by the Theorem (3.4), the problem has at least a nontrivial solution.

The existence and uniqueness of at least three positive solutions to FBVP (1) are obtained in the next result.

3.6. **Theorem:** Consider FBVP (1) and a continuous function h. Let c_1, c_2, c_3 and c_4 such that $0 < c_1 < c_2 < c_3 = c_2\,\gamma \leq c_4$ are constants. Assume that h satisfies the following hypothesis:

(D1): $h(t,y) < Xc_1,\ \forall\,(t,y) \in [0,1] \times [0,c_1]$,

(D2): $h(t,y) \geq Jbc_2\ \forall\,(t,y) \in [\zeta,1] \times [c_2,c_4]$,

(D3): $h(t,y) \leq Xc_3, \forall\,(t,y) \in [0,1] \times [0,c_4]$,

Thus, the FBVP (1) has at least three positive solutions $y_1, y_2, y_3 \in \mathcal{P}$ with $||y_i|| \leq c_4$, where

$i = 1,2,3$.

Proof: A nonnegative continuous concave functional Φ on a cone $\mathcal{P}$ is defined as follows,

$$\Phi(y) = \min_{t\in[\zeta,1]} |y(t)|.$$

Then $\Phi(y) \leq ||y||$. Assuming Θ and ϕ be two continuous convex functionals that are nonnegative on $\mathcal{P}$. They are defined as,

$$\Theta(y) = \phi(y) = ||y||,$$

and let ψ be a continuous, nonnegative function on cone $\mathcal{P}$, expressed as

$$\psi(y) = ||y||.$$

Then,

$$\psi(ly) = ||ly|| \leq |l|\,||y|| = |l|\,\psi(y) = l\,\psi(y), \qquad l \in [0,1].$$

$$\Phi(y) = \min_{t\in[\zeta,1]]}|y(t)| \leq ||y|| = \psi(y),$$

Additionally, we can find $\overline{U} \geq 1$ such that

$$||y|| = \phi(y) \leq \overline{U}\phi\,(y)\ \forall\, y \in \overline{\mathcal{P}\,(\phi,d)}.$$

Let the operator $T : \mathcal{P} \to \boldsymbol{B}$ defined in (3). $y(t)$ is a solution of the FBVP (1) only in the case that $y(t)$ is a fixed point of T in $\boldsymbol{B}$. Also, since for $t,s \in [0,1]$, $G(t,s) \geq 0$ by Lemma (3.2), and h: $[0,1] \times [0,\infty) \to [0,\infty)$ imply that $Ty(t) \geq 0$ for $0 \leq t \leq 1$. Now, we show the operator T: $\overline{\mathcal{P}(\phi,c_4)} \to \overline{\mathcal{P}(\phi,c_4)}$. Consider $y \in \overline{\mathcal{P}(\phi,c_4)}$. Thus $\phi(y) = \|y\| \leq c_4$ for $0 \leq y \leq c_4$ and $t \in [0,1]$. Then by (H3) and (4),

$$\begin{aligned} ||Ty|| = \max_{0\leq t\leq 1}|(T\,y)(t)| &= \max_{0\leq t\leq 1}\left|\int_0^1 G(t,s)\,h(s,y)ds\right| \\ &\leq c_4 X \max_{0\leq t\leq 1} \int_0^1 \frac{At^{\sigma-1}(1-s)^{\sigma-\rho-1}}{\Gamma(\sigma)}\,ds, \\ &\leq c_4(\sigma-\rho)\int_0^1 (1-s)^{\sigma-\rho-1}ds = c_4. \end{aligned}$$

Hence, operator T: $\overline{\mathcal{P}(\phi,c_4)} \to \overline{\mathcal{P}(\phi,c_4)}$. We now demonstrate that the operator T is completely continuous. As for $(t,s) \in [0,1] \times [0,1]$, $G(t,s)$ and $h(t,y)$ are continuous, signifies the operator T is continuous on a cone $\mathcal{P}$. For $c_4 > 0$, consider a set

$$\overline{\mathcal{P}_{c_4}} = \{y \in \mathcal{P} : ||y|| \leq c_4\},$$

Next, we are setting $L = \max_{\substack{0\leq t\leq 1,\\ 0\leq y\leq c_4}} h(s,y)$, then,

$$\begin{aligned} |(T\,y)(t)| &= \left|\int_0^1 G(t,s)h(s,y)ds\right| \\ &\leq L\int_0^1 \frac{At^{\sigma-1}(1-s)^{\sigma-\rho-1}}{\Gamma(\sigma)}ds, \\ &\leq \frac{LA}{\Gamma(\sigma)}\int_0^1 (1-s)^{\sigma-\rho-1}ds = \frac{L}{X} \end{aligned}$$

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implies that the operator T is uniformly bounded on $\overline{\mathcal{P}_{c_4}}$. Now since, on [0,1] × [0, 1], $G(t,s)$ is continuous, thus it is uniformly continuous there. Then, for every $\epsilon > 0$, there exists $\delta' > 0$ such that $|G(t_1,s) - G(t_2,s)| < \epsilon$ for any $y \in \overline{\mathcal{P}_{c_4}}$ and $t_1, t_2 \in [0, 1]$ with $|t_1 - t_2| < \delta'$,

$$|(T\,y)(t_1) - (T\,y)(t_2)| \le \int_0^1 |G(t_1,s) - G(t_2,s)|\, h(s,y)ds = \epsilon\, L,$$

which shows that $T(\overline{\mathcal{P}_{c_4}})$ is equicontinuous. Therefore, $T(\overline{\mathcal{P}_{c_4}})$ is relatively compact. Consequently, T is completely continuous in accordance with the application of the Arzel`a-Ascoli Theorem. Hence, on $\mathcal{P}$, for any convex function $\phi(y) = ||y||$, operator $T: \overline{\mathcal{P}(\phi, c_4)} \to \overline{\mathcal{P}(\phi, c_4)}$ is completely continuous.

Now the fact that the function $y(t) = c_3 = \frac{c_2 + c_4}{2} \in [0,1]$, where $c_3 \in \mathcal{P}\,(\phi, \Phi, \Theta, c_2, c_3, c_4)$ and $\Phi(\frac{c_2}{\gamma}) > c_2$, implies that

$\{y \in \mathcal{P}\ (\phi, \Phi, \Theta, c_2, c_3, c_4) \colon \Phi(y) > c_2\} \neq 0$. Thus for $\{y \in \mathcal{P}\,(\phi, \Phi, \Theta, c_2, c_3, c_4) : \Phi(y) > c_2\}$,

we have $c_2 \le y(t) \le c_3 = \frac{c_2}{\gamma}$ for $t \in [\zeta, 1]$.

Then by assumption (H2) and (4),

$$\Phi(T\,y) = \min_{t\in[\zeta,1]} \left|\int_0^1 G(t,s)\, h(s,y)ds\right|$$

$$\ge Jc_2 \min_{t\in[\zeta,1]} \int_\zeta^1 G(t,s)\, ds$$

$$\ge \quad Jc_2 \int_\zeta^1 \min_{t\in[\zeta,1]} \frac{\rho t^{\sigma-1} s(1-s)^{\sigma-\rho-1}}{\Gamma(\sigma)} ds = c_2.$$

This is satisfied $\forall\, \{y \in \mathcal{P}(\phi, \Phi, \Theta, c_2, c_3, c_4), \Phi(T\,y) > c_2\}$. Thus condition (B1) of Theorem (2.8) holds.

Next, we assume $y \in \mathcal{P}(\phi, \Phi, c_2, c_4)$ with $\Theta(y) > c_3 = \frac{c_2}{\gamma}$.

Thus,

$$\Phi(T\,y) = \min_{t\in[\zeta,1]} (T\,y)(t) \ge \gamma\, ||T\,y|| = \gamma\Theta(T\,y) > \gamma c_3 = c_2.$$

Hence, condition (B2) of Theorem (2.8) holds.

It is clear that $\phi(0) = 0 < c_1$ which imply $\phi \in \mathcal{R}(\phi, \psi, c_1, c_4)$. Now let $y \in \phi \in R(\phi, \psi, c_1, c_4)$ with $\psi(y) = ||y|| \le c_1$. Then by (H1) and (4),

$$\psi(T\,y) = \max_{t\in[0,1]} \left|\int_0^1 G(t,s)\, h(s,y)ds\right|$$

$$< Xc_1 \max_{t\in[0,1]} \int_0^1 G(t,s)ds$$

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$$< Xc_1 \max_{t\in[0,1]} \int_0^1 \frac{At^{\sigma-1}(1-s)^{\sigma-\rho-1}}{\Gamma(\sigma)} ds$$

$$\leq c_1.$$

Hence, condition (B3) of the Theorem (2.8) satisfied. Thus, by the Theorem (2.8), the problem (1) has at least three positive solutions y_1, y_2, and y_3, with $||y_i|| \leq c_4$, where $i = 1, 2, 3$, and $\psi(y_3) < c_1 < \psi(y_2)$, $\Phi(y_2) < c_2 < \Phi(y_1)$.
This completes proof of the Theorem (3.6).

3.7. **Example:** Assume that the given FBVP

$$D^{\frac{3}{2}} y(t) + h(t,y) = 0, \; y(0) = 0, \; D^{\frac{1}{2}}y(1) = D^{\frac{1}{2}}y\left(\frac{1}{4}\right), \quad t \in (0,1), \tag{7}$$

here, $\sigma = \frac{3}{2}, \beta = \frac{1}{2}, p = 1, \zeta = \frac{1}{4}$

$$h(t,y) = \begin{cases} \frac{(1+2t)}{10} y, & \forall\, t, y \in [0,1] \times [0,2], \\ 20(1+2t), & \forall\, t \in [0,1] \times \; y > 2. \end{cases} \tag{8}$$

By direct computation, we have $X \approx 0.886227$, $J \approx 7.56247$. We choose $c_1 = 1.2$, $c_2 = 2$, $c_3 = 65.5$, $c_4 = 70$, then,

$$h(t,y) = \frac{(1+2t)}{10} y \leq 0.36 < Xc_1 \approx 1.063472, (t,y) \in [0,1] \times [0,1.2],$$

$$h(t,y) = 20(1+2t) \geq 30 > Jc_2 \approx 15.12494, (t,y) \in \left[\frac{1}{4}, 1\right] \times [2,70],$$

$$h(t,y) = 20(1+2t) \leq 60 < Xc_4 \approx 62.03589, (t,y) \in [0,1] \times [0,70]$$

Thus, according to the Theorem (2.8), FBVP (7) has at least three positive solutions y_1, y_2, and y_3 with $\psi(y_3) < 1.2 < \psi(y_2)$, $\Phi(y_2) < 2 < \Phi(y_1)$.

1. Conclusion

In this study, we have conclusively found that there exist several positive solutions to FDE, using FPTs based on the Avery-Peterson FPT and the Schauder's FPT. We further illustrated the theoretical conclusions using instances. We conclude by stating that FPTs are an effective way to handle both ordinary differential equations and fractional differential equations.

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